#1. Let $N = 2015$, the expression becomes

$$N^2 - (N+1)(N-1) = N^2 - (N^2 - 1)$$
$$= N^2 - N^2 + 1$$
$$\boxed{= 1}$$

#2. The divisibility rule for 9 is the sum of the digits add up to 9.

Only $\boxed{10^{2015} + 8} = 1\underbrace{000 \ldots 0}_{2014\ 0\text{'s}}08$ meets the criterion.

#3. The sum of the first 12 numbers is

$$1 + 2 + \cdots + 12 = \frac{12 \cdot 13}{2} = 6 \cdot 13 = 78$$

Since each edge is part of 2 faces, the numbers on all the faces together must add up to $2 \cdot 78 = 156$.

Since there are 6 faces, each with the same sum, $\frac{156}{6} = \boxed{26}$ is the required sum.

BONUS: Find the way to do it.

#4. We want N such that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots + \frac{1}{N} < 3 < \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots + \frac{1}{N} + \frac{1}{N+1}$$

NOTE The "easy ones" $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = 1 + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right) + \frac{1}{4} + \frac{1}{5}$

$$= 1 + 1 + .25 + .2$$
$$= 2.45$$

So

$$2.45 + \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{N} < 3 < 2.45 < \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{N+1}$$

or

$$\frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{N} < .55 < \frac{1}{7} + \frac{1}{8} + \cdots + \frac{1}{N+1}$$

$\left.\begin{array}{l} \frac{1}{7} \sim .143 \\ \frac{1}{8} = .125 \\ \frac{1}{9} \sim .11 \\ \frac{1}{10} \sim .1 \end{array}\right\}$ these add up to about .47.

add $\frac{1}{11} \sim .09$, will make the sum $> .55$. So $N = 10$, and the answer is $\boxed{11}$

#5 101 is prime

$102 = 2\cdot 3\cdot 17$ is square-free

103 is prime

$104 = 8\cdot 13 = 2^3\cdot 13$ is not square free

$105 = 5\cdot 21 = 5\cdot 3\cdot 7$ is square free

So $\boxed{102}$ is the least.

#6 $\dfrac{2015}{2+0+1+5} = \dfrac{2015}{8} = 258\frac{7}{8}$, so $\boxed{\frac{1}{8}}$ adds to an integer.

#7

$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 64$
$2^7 = 128$
$2^8 = 256$
$\vdots$

2, 4, 8, 6 repeats

2^{4k} ends in 6, so 2^{2012} ends in 6, and 2^{2015} ends in 8.

$3^1 = 3$
$3^2 = 9$
$3^3 = 27$
$3^4 = 81$
$3^5 = 243$
$3^6 = 729$
$3^7 = 2187$
$3^8 = 6561$
$\vdots$

3, 9, 7, 1 repeats

3^{4k} ends in 1

so 2^{2012} ends in 1

and 2^{2015} end in 7.

5^{2015} ends in 5 since all power of 5 end in 5.

So $8 + 7 + 5 = 20$ ends in $\boxed{0}$.

#8 Exterior angles add up to 360, in either direction.

So the sum $p+q+r+s+t+u+v+w+x+y =$

$(p+r+t+v+x) + (q+s+u+w+y) =$

$360 + 360 \boxed{= 720}$

#9 The triangles both have height 3. Let their bases be b_1 and b_2

then $\frac{1}{2}b_1\cdot 3 + \frac{1}{2}b_2\cdot 3 = \frac{1}{2}(b_1+b_2)\cdot 3$

$= \frac{1}{2}(14)\cdot 3$

$\boxed{= 21}$

#10 There will be $5\cdot4\cdot3\cdot2\cdot1$ permutations. $5\cdot4\cdot3\cdot2\cdot1 = 120$.
It is best to work backwards since the required USAMO is near the end.

120	USOMA
119	USOAM
118	USMOA
117	USMAO
116	USAOM
$\boxed{115}$	USAMO

#11

2 0 15 0 2 0 5 0

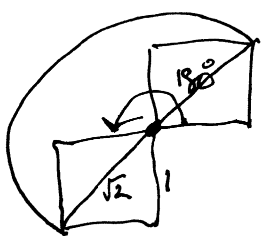
0 0 0 0 1 2 2 5 5

MODE = 4. There are 4 0's
MEDIAN = 1. Half are above 1, half are below 1
MEAN = $\frac{15}{9} = 1\frac{2}{3}$ so $\boxed{\text{mean} > \text{median} > \text{mode}}$

#12

$2015 = 5^1\cdot13^1\cdot31^1$, so it has $(1+1)(1+1)(1+1) = 8$ divisors
But divisor 2015 is bigger than 500, so only 7 of the divisors are less than 500. $\boxed{\frac{7}{500}}$

#13. The region swept out looks like this



It is a half circle of radius $\sqrt{2}$, and two half squares.
So the area is

$\frac{1}{2}\pi(\sqrt{2})^2 + 2\left(\frac{1}{2}1^2\right)$
$= \boxed{\pi + 1}$

#14 There are six months that use only 2, 0, 1, or 5.

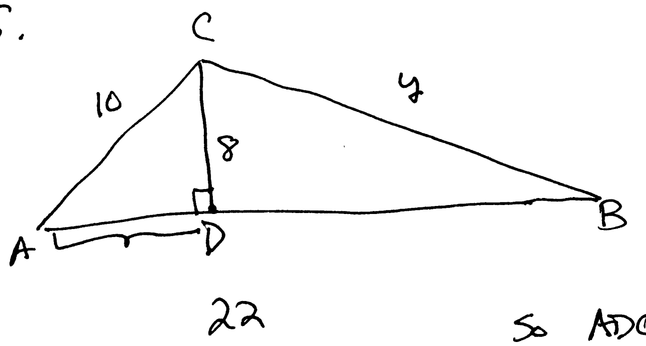
01, 02, 05, 10, 11, 12

There are eleven days that use only 2, 0, 1, 5.

1, 2, 5, 10, 11, 12, 15, 20, 21, 22, 25

$6 \cdot 11 = \boxed{66}$

#15.



If the area is 88, then taking $AB = 22$.

$\frac{1}{2} 22 \cdot h = \frac{1}{2} 22 \cdot CD = 88$

so $h = CD = 8$

So ADC is a 6-8-10 right triangle, and $AD = 6$.

That make $BD = 22 - 6 = 16$.

By the pythagorean theorem $16^2 + 8^2 = y^2$, so

$$y^2 = 16^2 + 8^2 = 8^2(2^2+1) = 8^2 \cdot 5$$

$$y = \boxed{8\sqrt{5}}$$

#16 ~~(6+7+8-9-10-12~~

$(6+7+8-9-10) = 21-19 = 2$

$(11+12+13-14-15) = 36-29 = 7$

...

$(2006+2007+2008-2009-2010) = 2002$

$(2011+2012+2013-2014-2015) = 2007$

$2 + 7 + 12 + \cdots + 2002 + 2007$ This is an arithmetic progression with common difference 5. First term is 2. Last term is 2007.

$(1 \cdot 5 - 3) + (2 \cdot 5 - 3) + (3 \cdot 5 - 3) + \cdots + (401 \cdot 5 - 3) + (402 \cdot 5 - 3)$

So there are 402 terms. The sum is then

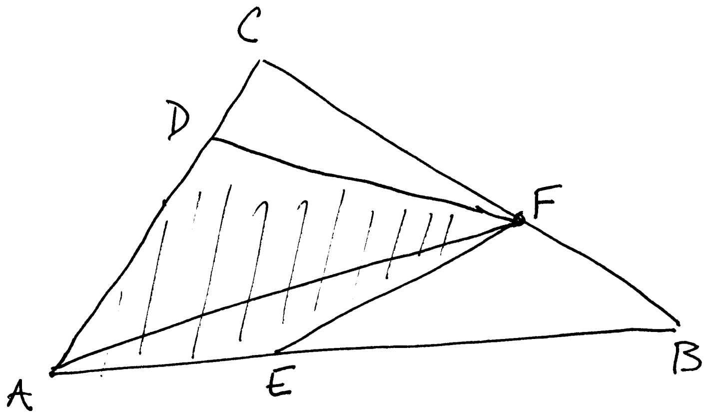
$$\frac{402}{2}(2+2007) = 201 \cdot 2009 = \boxed{403,809}$$

#17 $3\frac{1}{2}$ hours has 210 minutes, so wen must goof at most less than 105 minutes.

He currently goofs $\frac{6}{7} \cdot 210 = 180$ minutes.

$180 - 105 = 75$. So he must goof less by at least $\boxed{76}$ minutes to be strictly less than 105.

#18



Draw line AF.

Triangles with same height have areas in proportion to their bases.

So AFC is $\frac{3}{4}$ ABC and then CDF is $\frac{1}{4}$ AFC.

and AFB is $\frac{1}{4}$ ABC and BEF is $\frac{3}{4}$ AFB.

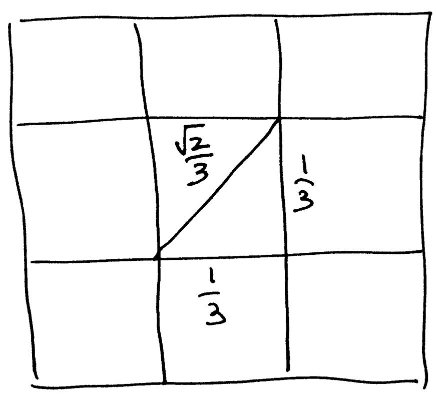
So $CDF = \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{16}$ ABC and

$BEF = \frac{1}{4} \cdot \frac{3}{4} = \frac{3}{16}$ ABC.

So $BEF + CDF = 2 \cdot \frac{3}{16} = \frac{3}{8}$ of ABC

So the shaded region is $\boxed{\frac{5}{8}}$ of ABC.

#19 Pidgeon Hole Principle.



Draw a 3x3 grid in the square. Each small square is $\frac{1}{3} \times \frac{1}{3}$.

If I pick 10 points, then 2 of them must be in one of the small squares, at most $\boxed{\frac{\sqrt{2}}{3}}$ apart.

#20 Esther has earned

$$\begin{array}{r} 800 \\ 750 \\ 780 \\ 800 \\ +760 \\ \hline 3890 \end{array}$$

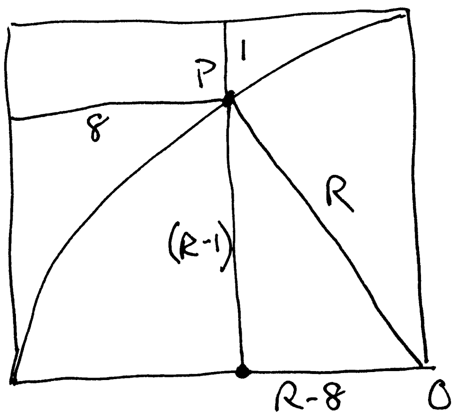
3890 points

For an average of $3890/5 = 778$. So she wants an average of 788 for 9 exams, which means she needs $788 \times 9 = 7092$ points.

But $7092 - 3890 = 3202 > 4 \cdot 800$.

So Esther can't meet her goal.

#21.



let R = the side of the square

OP = R.

By the pythagorean theorem

$$(R-8)^2 + (R-1)^2 = R^2$$

$$\cancel{R^2} - 16R + 64 + R^2 - 2R + 1 = \cancel{R^2}$$

$$R^2 - 18R + 65 = 0$$

$$(R-5)(R-13) = 0$$

$$R = \boxed{13}$$

#22 Multiply top and bottom by 5^{2015} to get

$$\frac{2015 \cdot 5^{2015}}{2^{2015} \cdot 5^{2015}} = \frac{2015 \cdot 5^{2015}}{10^{2015}}$$

Dividing by powers of ten doesn't affect the order of digits

What are the last digits of $2015 \cdot 5^{2015} = 403 \cdot 5^{2016}$

Powers of 5:

5

25

$5^3 = 125$

$5^4 = 625$

$5^5 = 3125$

15625

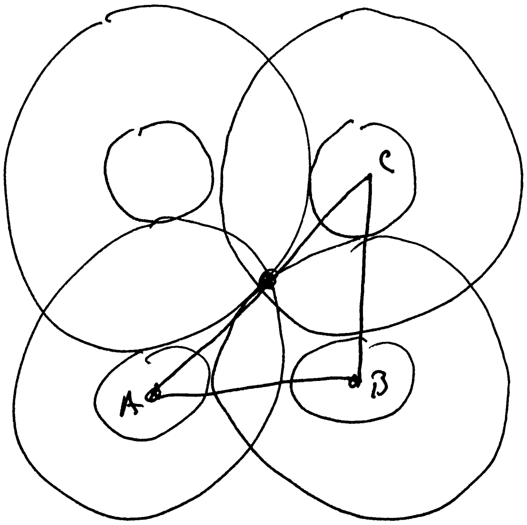
78125

REPEATS LAST 3 DIGITS, so 2^{2016} ends in 625

$$\begin{array}{r} \ldots\ldots 625 \\ \times \quad 403 \\ \hline 875 \\ \ldots 5000 \\ + \qquad \\ \hline 875 \end{array}$$

So the third digit from the end is $\boxed{8}$

#23

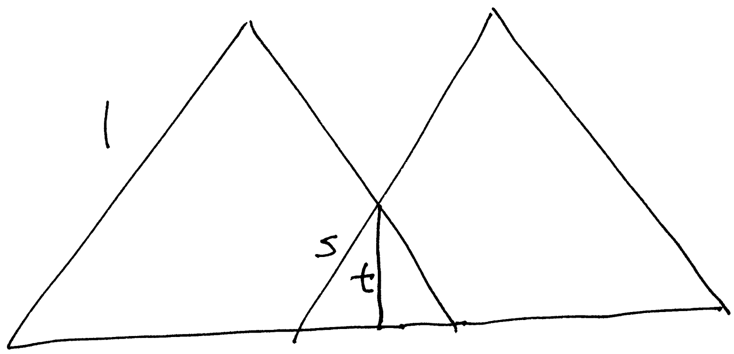


$AC = 2$

$AB = BC = 1+R$, where R is the radius of the smaller circle

$$(1+R)^2 + (1+R)^2 = 2^2$$
$$2(1+R)^2 = 2^2$$
$$(1+R)^2 = 2$$
$$R+1 = \sqrt{2}$$
$$R = \boxed{\sqrt{2}-1}$$

#24



let s be the side of the smaller triangle

The area of an equilateral triangle of side s is $\frac{s^2\sqrt{3}}{4}$, so

$$\frac{s^2\sqrt{3}}{4} = \frac{1}{11}\left(\frac{1^2\cdot\sqrt{3}}{4} + \frac{1^2\sqrt{3}}{4} - \frac{s^2\sqrt{3}}{4}\right)$$

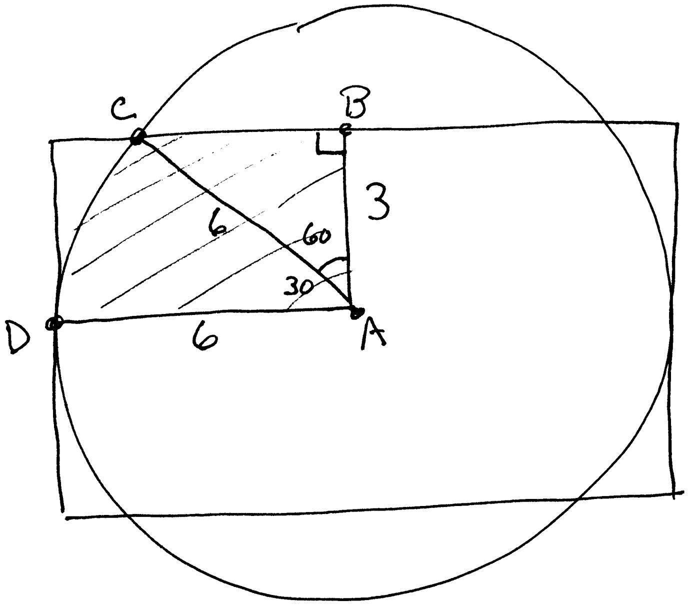
$$11\cdot\frac{s^2\sqrt{3}}{4} = 2\cdot\frac{\sqrt{3}}{4} - \frac{s^2\sqrt{3}}{4}$$

$$\frac{12s^2\sqrt{3}}{4} = \frac{2\sqrt{3}}{4} \Rightarrow s^2 = \frac{1}{6} \text{ so } s = \sqrt{\frac{1}{6}} = \frac{\sqrt{6}}{6}$$

The altitude of an equilateral triangle of side s is $\frac{s\sqrt{3}}{2}$

So the altitude t is $\frac{s\sqrt{3}}{2} = \frac{\sqrt{6}}{6}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{18}}{12} = \frac{3\sqrt{2}}{12} = \boxed{\frac{\sqrt{2}}{4}}$

#25



Since A is the center, AB = 3, and AC = 6. So ABC must be a 30-60-90 triangle with the side ratio $1:\sqrt{3}:2$.

The shaded region is made up of 4 of the above shaded area. ABC is a 30-60-90 right triangle so $BC = 3\sqrt{3}$. The area of the sector $A\widehat{CD}$ is $\frac{30}{360}\cdot(\pi 6^2) = \frac{1}{12}\cdot\pi\cdot 6^2 = \frac{36\pi}{12} = 3\pi$.

The area of triangle ABC is $\frac{1}{2}\,3\sqrt{3}\cdot 3 = \frac{9}{2}\sqrt{3}$

The area of the whole shaded region is then

$$4\left(3\pi + \frac{9}{2}\sqrt{3}\right) = \boxed{12\pi + 18\sqrt{3}}$$