Mock AMC 8 2014 Answer Key

1. C Arithmetic and Operations (Order of Operations – PEMDAS)
2. C Arithmetic and Operations (Fractions/Decimals)
3. C Arithmetic and Operations (Division)
4. C Arithmetic and Operations (Calendar)
5. D Number Theory - Pattern Recognition
6. D Number Theory - Factorization
7. B Number Theory – Number of Factors Formula
8. D Number Theory – Sum of Factors Formula
9. E Number Theory – Pattern Recognition (Squares)
10. B Number Theory - Logic
11. E Arithmetic and Operations (Fractions)
12. D Word Problems - Work
13. A Geometry – Angle Chasing
14. A Geometry – Pythagorean Theorem
15. A Geometry – Pythagorean Theorem
16. E Geometry – Special Triangles 30-60-90
17. A Geometry – Similar Triangles/Area Relationships
18. C Counting – Inclusion-Exclusion Principal (Hard)
19. C Number Theory - LCM, GCM and formula
20. D Probability - Dice
21. A Geometry – Angle Chasing
22. D Geometry – Logic/Circles/Special Triangles 45-45-90
23. E Probability - Dice
24. B Counting – Casework (Hard)
25. B Counting – Casework (Very Hard)
26. E Probability – Inclusion-Exclusion Principal (Very Hard)

Some Additional Topics Not Covered to think about:

1. Arithmetic and Operations (Percents)
2. Geometry – Circle Area and Perimeter
3. Geometry – Area Relationships
4. Geometry – Triangle Inequality
5. Counting - Permutations/Combinations
6. Prealgebra - Mean/Median/Mode/Range
7. Word Problems – Rate-Time-Distance
8. Word Problems - Mixture
9. Solid Geometry – Vertices/Edges/Faces/Euler's Formula
10. Number Theory – Divisibility Rules 2,3,5,6,8,9,10,11,12
11. Number Theory – Terminating Digits

#1) Use order of operations, multiplication first, left to right.

PEMDAS = Parenthesis, Exponents, Multiplication, Division, Addition, Subtraction

There are 6 possibilities:

$$2+0\times1-4 = 2+(0\times1)-4 = 2-4=-2$$
$$2+0-1\times4 = 2+0-(1\times4) = 2-4=-2$$
$$2\times0+1-4 = (2\times0)+1-4 = 0-3=-3$$
$$2\times0-1+4 = (2\times0)-1+4 = 0+3=3$$
$$2-0\times1+4 = 2-(0\times1)+4 = 2+4=6$$
$$2-0+1\times4 = 2-0+(1\times4) = 2+4=6$$

There are 4 unique values.

#2) It's easier to see what is going on if we convert to fractions:

$$0.1\times0.2\times0.3\times0.4\times\square = 0.12$$
$$\frac{1}{10}\times\frac{2}{10}\times\frac{3}{10}\times\frac{4}{10}\times\square = \frac{12}{100}$$
$$\text{or}\quad \frac{24}{10000}\cdot\square = \frac{12}{100}$$
$$\text{or}\quad \square = \frac{12}{100}\cdot\frac{10000}{24} = \frac{100}{2} = 50$$

#3) $$\frac{2014}{2+0+1+4} = \frac{2014}{7} = 287 \text{ Remainder } 5$$
$$= 287\tfrac{5}{7} \text{ which is closest to } 288.$$

#4) For the moment, let a year be 365 days, then 5 years is 1825 days and 2014 days is less than 6 years or 2190 days. So 2014 is 5 years (60 months) and this leaves 2014 − 1825 = 189 days to consider. The months have the following number of days:

January	31	July	31
February	28	August	31
March	31	September	30
April	30	October	31
May	31	November	30
June	30	December	31

The sum of the number of days of the first 6 months is 181, so 189 days puts us into July. So Esther is 66 months old. (60+6). What about leap years? Esther is only 5, so she could only have seen at most 2 leap years and this would not affect the math.

#5) $$\underbrace{1-2}_{-1}+\underbrace{3-4}_{-1}+\underbrace{5-6}_{-1}+\cdots+\underbrace{2013-2014}_{-1}$$

If we pair the differences, each pair is −1.

There are 1007 pairs. So the series adds up to −1007.

#6) 2014 factors as $2014 = 2\cdot 19\cdot 53$

but this is only 3 factors, the problem says 4!?

The trick is to multiply by 1, which doesn't change the product.

$$2014 = 1\cdot 2\cdot 19\cdot 53$$

and $1+2+19+53 = 75$

#7) 2013 factors as $3\cdot 11\cdot 61$

2014 factors as $2\cdot 19\cdot 53$

2015 factors as $5\cdot 13\cdot 31$ so

$$2013\times 2014\times 2015 = 2\cdot 3\cdot 5\cdot 13\cdot 11\cdot 19\cdot 31\cdot 53\cdot 61$$

The even factors would include the 2, odd factors without the 2.

To find the number of divisors of a number n:

1. Write n as a product of prime powers (prime factorization)
2. Add 1 to each of the exponents of the primes
3. Multiply these values together

If $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, then n has $(e_1+1)(e_2+1)\cdots(e_k+1)$ factors.

$2013\times 2014\times 2015$ has 9 prime factors, all with exponent 1. So the number of factors will be $(1+1)(1+1)\cdots(1+1) = 2^9 = 512$.

Half of the factors will be even - why? So the answer is 256.

#8 To find the sum of the divisors of a number n:

1. Write n as a product of prime powers (prime factorization)
2. Compute the sum of all powers of each prime (from $\emptyset$ up to the exponent inclusive)
3. Multiply these sums together.

If $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, then the sum of the divisors is

$$(1+p_1+p_1^2+\cdots+p_1^{e_1})(1+p_2+p_2^2+\cdots+p_2^{e_2})\cdots\cdots(1+p_k+p_k^2+\cdots+p_k^{e_k})$$

which by using our formula for a geometric series is

$$\left(\frac{p_1^{e_1+1}-1}{p_1-1}\right)\left(\frac{p_2^{e_2+1}-1}{p_2-1}\right)\cdots\left(\frac{p_k^{e_k+1}-1}{p_k-1}\right)$$

This all looks pretty ugly, but in most cases it simplifies

$$2014 = 2\cdot 19\cdot 53 = 2^1\cdot 19^1\cdot 53^1$$

So the sum formula reduces to $(1+2^1)(1+19^1)(1+53^1)$

$$= 3\cdot 20\cdot 54 = 60\cdot 54 = 3240$$

#9 Notice the last number of each row is a perfect square!

Now $44^2 = 1936$ and $45^2 = 2025$, so

2014 will be in the row ending with 45^2, row 45

#10) The way to do these problems is to assume each answer and then see if you get a contradiction.

CASE A) A TRUE, SO M IS ODD
D MUST BE FALSE, SO N IS EVEN
BUT THEN $M-N = \text{ODD} - \text{EVEN} = \text{ODD}$ IS TRUE, SO C WOULD ALSO BE TRUE.
CONTRADICTION

CASE B) B TRUE, SO N^2 IS EVEN, SO N IS EVEN
A MUST BE FALSE, SO M IS ALSO EVEN
THEN C IS FALSE SINCE $M-N = \text{EVEN} - \text{EVEN}$
THEN D IS FALSE SINCE N IS EVEN
AND E IS FALSE SINCE M, N have At least 2 as a COMMON FACTOR.

WE ARE DONE – ANSWER IS B.

OTHER CASES:

CASE C) C TRUE, SO M−N IS ODD
A MUST BE FALSE, SO M IS EVEN
D MUST BE FALSE, SO N IS EVEN
BUT THEN M−N IS EVEN, CONTRADICTING OUR ASSUMPTION.

CASE D) D TRUE, SO N IS ODD
A MUST BE FALSE, SO M IS EVEN
BUT THEN $M-N = \text{EVEN} - \text{ODD} = \text{ODD}$ IS TRUE, SO C WOULD ALSO BE TRUE
CONTRADICTION

CASE E) M AND N HAVE NO COMMON FACTORS
A MUST BE FALSE, SO M IS EVEN
D MUST BE FALSE, SO N IS EVEN
BUT THEN M AND N HAVE AT LEAST A COMMON FACTOR 2.

#11) WRITE EACH FRACTION AS AN EXPRESSION INVOLVING 1.

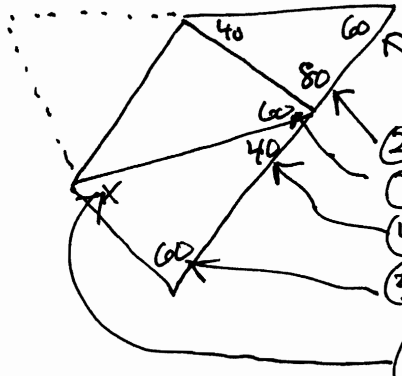
$$\frac{2013}{2014} = 1 - \frac{1}{2014} \qquad \frac{1006}{1007} = 1 - \frac{1}{1007} \qquad \frac{1007}{1006} = 1 + \frac{1}{1006}$$

$$\frac{2014}{2013} = 1 + \frac{1}{2013} \qquad \frac{2014}{2015} = 1 - \frac{1}{2015}$$

$\frac{1}{2015}$ IS THE SMALLEST FRACTION THAT WE ARE ADDING OR SUBTRACTING.

#12) FOR WORK PROBLEMS A COMMON TECHNIQUE IS TO FIND A "UNIT VALUE" – IN OTHER WORDS A RELATION INVOLVING 1 UNIT OF WORK, OR TIME, OR PEOPLE.
So if 19 people do something in 4 minuts the $19 \times 4 = 76$ people will do the same thing is $4/4 = 1$ minute. So 76 people can eat 53 packages in 1 minute. Now we know $53 \times 38 = 2014$, So 76 people will eat 2014 $(= 53 \times 38)$ packages in 1×38 minutes.

#13) This is called "angle chasing"



① This is 60 since the triangle is equilateral

② This is 80 since the angles of a triangle add up to 180.

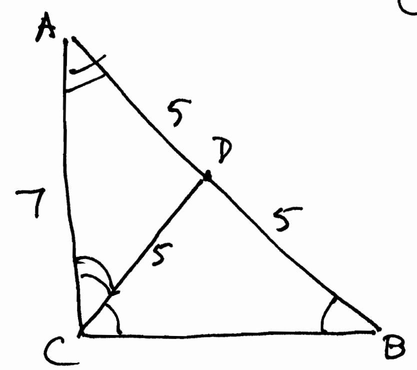
③ This is also 60 since its the angle of the original triangle

④ This is then 40 since angles on a line add up to 180

⑤ This is 60 since the triangle is equilateral

⑥ So $x = 180 - 40 - 60 = 80$ since the angles of a triangle add up to 180.

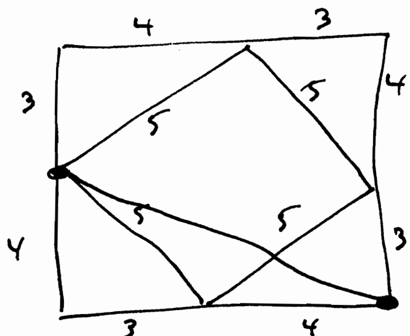
#14)



From the problem ΔBDC is isosceles, so base angles are equal as marked (.
Similarly, the base angles of ΔADC are also equal, marked with ((.
So $2((+ (() = 180$ since those are the angles of ΔACB, so $(+ ((= 90°$ and $\angle ACB = 90°$.

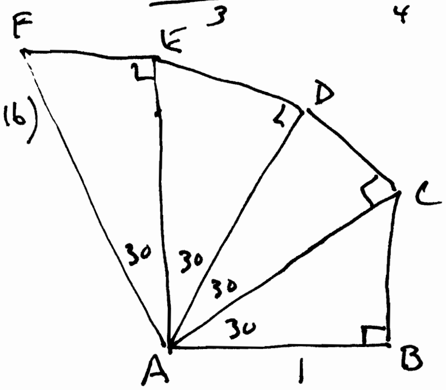
So we have a right triangle, and $BC^2 + AC^2 = AB^2$ by the Pythagorean Theorem, and $BC^2 + 7^2 = 10^2 \Rightarrow BC^2 = 10^2 - 7^2 = 100 - 49$. So $BC = \sqrt{51}$

#15)



We have 3-4-5 right triangles here. So the longest distance is the hypotenuse of a right triangle with legs 7 and 4. $\sqrt{7^2+4^2} = \sqrt{65}$
How do we know it's 3-4-5? Suppose we have x, y, then $x+y=7$ and $x^2+y^2=5 \Rightarrow xy = 12$, so x and y are roots of $x^2 - 7x + 12 = (x-3)(x-4) = 0$.

#16)



It's easy to get confused here. You <u>DON'T</u> want to use the pythagorean theorem. Use the ratio in the special triangle 30-60-90. The ratio is $2 : \sqrt{3} : 1$.

So $\frac{AC}{AB} = \frac{2}{\sqrt{3}}$ $\frac{AD}{AC} = \frac{2}{\sqrt{3}}$ $\frac{AE}{AD} = \frac{2}{\sqrt{3}}$ $\frac{AF}{AE} = \frac{2}{\sqrt{3}}$

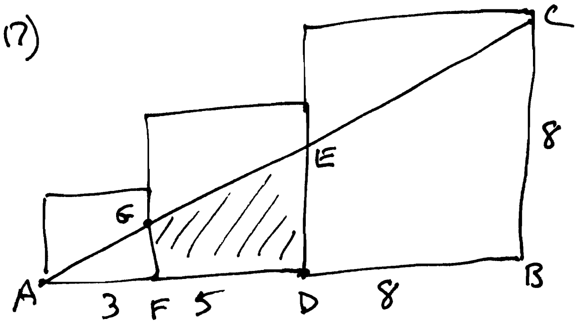
Multiplying, gives

$$\frac{AF}{\cancel{AE}} \cdot \frac{\cancel{AE}}{\cancel{AD}} \cdot \frac{\cancel{AD}}{\cancel{AC}} \cdot \frac{\cancel{AC}}{AB} = \frac{2}{\sqrt{3}} \cdot \frac{2}{\sqrt{3}} \cdot \frac{2}{\sqrt{3}} \cdot \frac{2}{\sqrt{3}}$$

$$\frac{AF}{AB} = \left(\frac{2}{\sqrt{3}}\right)^4 = \frac{16}{9}$$

$$AF = \frac{16}{9}$$

#17)



BC = 8 since we have squares, so the area of ABC $= \frac{1}{2}\cdot 16\cdot 8 = 64$

Since $\frac{AD}{AB} = \frac{8}{16} = \frac{1}{2}$, the area of $\frac{\Delta ADE}{\Delta ABC} = \frac{1}{4}$

so the area of $\Delta ADE = 16$. Similarly
$(= \frac{1}{4}\cdot 64)$

Since $\frac{AF}{AD} = \frac{3}{8} \Rightarrow \frac{\text{Area of } \Delta AFG}{\text{Area of } \Delta ADE} = \frac{9}{64}$, so the area of $\Delta AFG = \frac{9}{64}\cdot 16 = \frac{9}{4}$

then the area of quadrilateral FDEG is $16 - \frac{9}{4} = \frac{64}{4} - \frac{9}{4} = \frac{55}{4}$.

#18) We know that $44^2 = 1936$ and $45^2 = 2025$, so 44 squares are removed from the sequence $1, 2, 3, 4, \ldots, 2014$. Also $12^3 = 1828$ and $13^3 = 2197$, so 12 cubes are removed from the sequence. But some numbers are both squares and cubes, 6th powers. $1^6 = 1$, $2^6 = 32$, $3^6 = 729$, $4^6 = 4096$, so 3 numbers have been removed twice and we add them back – inclusion/exclusion. So we're removed $44 + 12 - 3 = 53$ numbers, so the 2014th number would be $2014 + 53 = 2067$, BUT $45^2 = 2025 < 2067$, so it would be removed too, so the answer is 2068.

#19) METHOD 1). The LCM of two numbers is the number formed by taking the highest powers in the numbers prime factorizations.

RECOGNIZE $3125 = 5^5$, $256 = 2^8$

$312500 = 5^5\cdot 2^2\cdot 5^2 = 2^2\cdot 5^7$

$25600 = 2^8\cdot 2^2\cdot 5^2 = 2^{10}\cdot 5^2$

$LCM(312500, 25600) = 2^{10}\cdot 5^7 = 2^3\cdot 2^7\cdot 5^7$
$= 2^3\cdot (10)^7$
$= 8 \cdot 10{,}000{,}000$
$= 80{,}000{,}000$

METHOD 2) Use the formula

(LCM)(GCD) = PRODUCT

The GCD of 312500 and 25600 is obviously 100. why?

So $LCM \cdot 100 = 312500 \cdot 25600$

$LCM \cdot 100 = 5^5 \cdot 100 \cdot 2^8 \cdot 100$

$LCM = 2^8\cdot 5^5\cdot 100$
$= 8\cdot 2^5\cdot 5^5\cdot 100$
$= 8\cdot 100{,}000 \cdot 100$
$= 8\cdot 10{,}000{,}000$
$= 80{,}000{,}000$

#20) The possibilities - "sample space" are best represented as a 6x6 table for multiplication

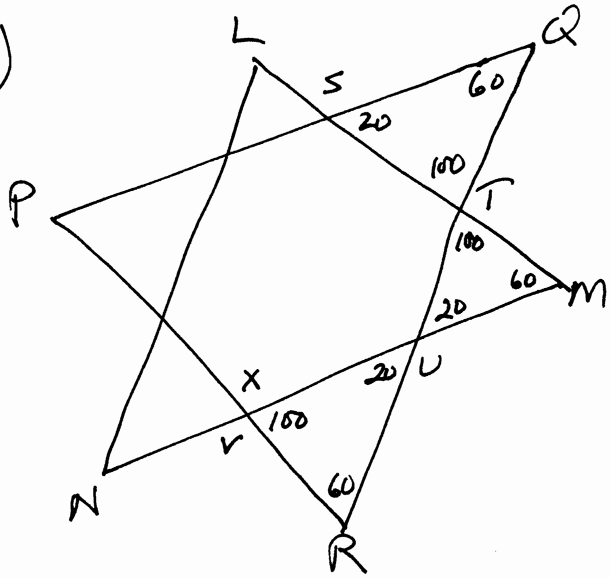
x	-3	-2	-1	0	1	2
-3	9	6	3	0	-3	-6
-2	6	4	2	0	-2	-4
-1	3	2	1	0	-1	-2
0	0	0	0	0	0	0
1	-3	-2	-1	0	1	2
2	-6	-4	-2	0	2	4

There are 36 possibilities with 2 dice.

Our "event space" - the things we want are boxed. There are 9 possibilities for the product to be positive with negative x negative, and 4 possibilities for the product to be positive with positive x positive. $9+4=13$

So the probability is $13/36$.

#21)



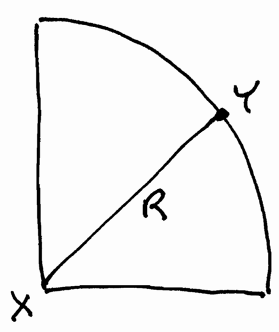
Label the extra points S, T, U, V.

1. $\angle SQT = 60$ since PQR is equilateral
2. $\angle STQ = 100$ since angles add up to 180
3. $\angle UTM = 100$ by vertical angles
4. $\angle M = 60$ since LMN is equilateral
5. $MUT = 180 - 100 - 60 = 20$ since angles add up to 180
6. $\angle VUR = 20$ by vertical angles
7. $\angle R = 60$ since PQR is equilateral
8. $\angle UVR = 100$ since angles add up to 180

So 9. $\angle UVP = X = 80$ by supplementary angles.

#22)

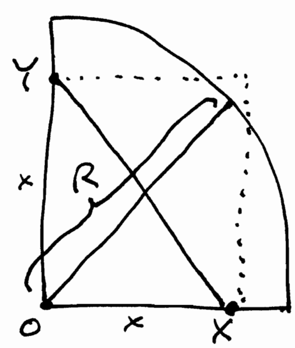
A



$XY = R$

XY is the radius

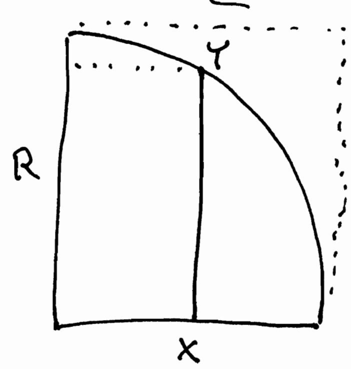
B



$XY > R$

XY is the diagonal of a square with other diagonal (=XY) clearly larger than R.

C



$XY < R$

XY is clearly smaller than the side of the enclosing square.

So $C < A < B$.

For B, letting $OX = OY = x$. $\frac{1}{2}x^2 = \frac{1}{2}\cdot\frac{1}{4}\pi R^2$, so $x^2 = \frac{\pi R^2}{4}$ and $x = \sqrt{\frac{\pi}{4}}R$

so the diagonal $XY = x\sqrt{2} = \sqrt{\frac{2\pi}{4}}R = \sqrt{\frac{\pi}{2}}R > R$.

#23) The possibilities – "sample space" are best represented as a 6x6 addition table

+	-3	-2	-1	0	1	2
-3	-6	-5	-4	-3	-2	-1
-2	-5	-4	-3	-2	-1	0
-1	-4	-3	-2	-1	0	1
0	-3	-2	-1	0	1	2
1	-2	-1	0	1	2	3
2	-1	0	1	2	3	4

There are 36 possibilities with 2 dice. Our "event space" the things we want – a non-negative, (including $\emptyset$) sum, are boxed with 15 possibilities

$$\frac{15}{36} = \frac{5}{12}$$

#24) Note that there is only 1 way to go from a C to the 8.

This is a counting problem with casework

The best approach is to look at the M's.

For each of the 4 "corner" Ms, there are 5 A's to get to it from. and 1 C that is can go to $A \to M \to C \to 8$

$5 \cdot 1 \cdot 1 \cdot 1 = 5$ ways

There are 4 corners, so this gives $4 \times 5 = 20$ paths

For each of the 4 "side" center Ms, there are 3 A's that can go to it and 3 Cs that it can go to. $A \to M \to C \to 8$

$3 \to 1 \to 3 \to 1 = 9$ ways

There are 4 center Ms, so this gives $4 \times 3 \times 3 = 36$ paths.

For the rest of the 8 Ms, there are 3 A's that neighbor it, and 2 Cs that neighbor it. $A \to M \to C \to 8$

$3 \cdot 1 \cdot 2 \cdot 1 = 6$ ways

There are 8 of these Ms, so this gives $8 \times 3 \times 2 = 48$ paths

$20 + 36 + 48 = 104$ total paths.

#25) This is a very famous problem. For a general stack of N pancakes it is still an unsolved problem. For small values of N you can list out the possibilities.

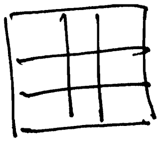
Suppose the pancakes are numbered 1, 2, 3, 4 with 1 being the smallest and 4 being the largest. We want to end up with the stack TOP → BOTTOM being 1 2 3 4. The pancakes to be flipped with each step will be underlined.

There are 4! = 24 possible starting positions, the number of flips needed is circled under the sequence

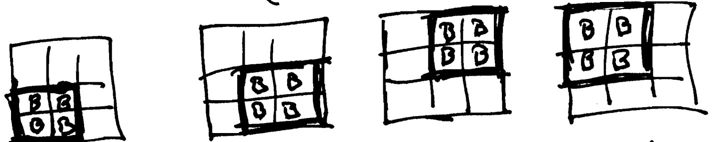
1234 ∅	1243 3421 4321 1234 3	1342 4312 2134 1234 3	1324 2314 3214 1234 3	1423 4123 3214 1234 3	1432 2341 4321 1234 3
2134 1234 1	2143 4123 3214 1234 3	2314 3214 1234 2	2341 4321 1234 2	2413 4213 3124 2134 1234 4 (boxed)	2431 3421 4321 1234 3
3124 2134 1234 2	3142 1342 4312 2134 1234 4 (boxed)	3214 1234 1	3241 2341 4321 1234 3	3412 4312 2134 1234 3	3421 4321 1234 2
4123 3214 1234 2	4132 2314 3214 1234 3	4231 2431 3421 4321 1234 4 (boxed)	4213 3124 2134 1234 3	4312 2134 1234 2	4321 1234 1

The 3 boxed starting positions require 4 FLIPS

26) This is another (more complicated) example of the principal of inclusion – exclusion



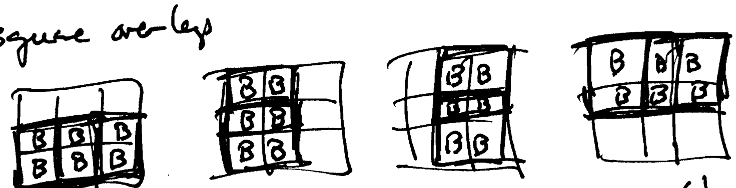
3×3 grid there are 4 possibilities for 2×2 squares



The probability that a 2×2 square is blue is $\left(\frac{1}{2}\right)^4 = \frac{1}{2^4}$ and there are 4 possibilities, so $4 \times \frac{1}{2^4} = \frac{4}{2^4}$, but...

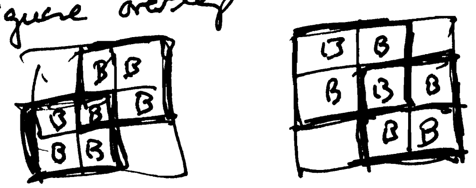
~~But~~ we have to subtract off the probability that 2 of these are simultaneously blue, which would be

case 1) 2 square overlap



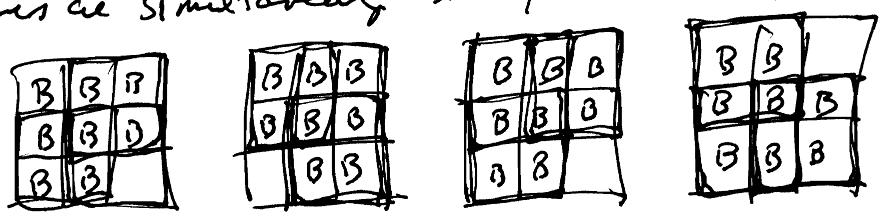
this can happen in 4 ways, so $4 \times \left(\frac{1}{2^6}\right) = \frac{4}{2^6}$

case 2) 1 square overlap



this can happen in 2 ways, so $2 \times \frac{1}{2^7} = \frac{2}{2^7}$

but, we have to ADD BACK the probability that 3 2×2 squares are simultaneously blue, which can happen in 4 ways:



or $4 \times \left(\frac{1}{2}\right)^8 = \frac{4}{2^8}$

but we have to SUBTRACT OFF the probability that 4 2×2 squares are simultaneously blue which can only happen in 1 way with all squares blue. So the final value is

$$4 \cdot \frac{1}{2^4} - \left(4 \cdot \frac{1}{2^6} + 2 \cdot \frac{1}{2^7}\right) + 4 \cdot \frac{1}{2^8} - \frac{1}{2^9} \boxed{= \frac{95}{512}}$$