MAA AMC

American Mathematics Competitions

Official Solutions

MAA American Mathematics Competitions

39th Annual

AMC 8

Thursday, January 18, 2024 through Wednesday, January 24, 2024

These official solutions give at least one solution for each problem on this year's competition and show that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

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The problems and solutions for this AMC 8 were prepared by the MAA AMC 8 Editorial Board under the direction of:

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1. What is the ones digit of

$$222{,}222 - 22{,}222 - 2{,}222 - 222 - 22 - 2\,?$$

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

Answer (B): The difference $222{,}222-22{,}222$ has a ones digit (also called the units digit) of 0. Subtracting 2,222 from that quantity results in a number whose units digit is 8. Continuing in this way, the subsequent differences will have units digits of 6, then 4, and finally 2.

OR

The units digit of the result will depend only on the units digits of the numbers in the expression. The problem is equivalent to finding the units digit of

$$222{,}222 - 2 - 2 - 2 - 2 - 2.$$

Subtracting $5 \cdot 2 = 10$ from 222,222 gives a units digit of 2.

2. What is the value of this expression in decimal form?

$$\frac{44}{11} + \frac{110}{44} + \frac{44}{1100}$$

(A) 6.4 **(B)** 6.504 **(C)** 6.54 **(D)** 6.9 **(E)** 6.94

Answer (C): Note that

$$\frac{44}{11} = 4 \qquad \text{and} \qquad \frac{11}{44} = \frac{1}{4},$$

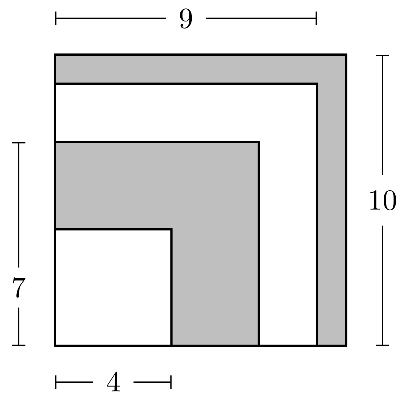
and

$$\frac{110}{44} = \frac{11}{44} \cdot 10 = \frac{1}{4} \cdot 10 = 2.5 \qquad \text{and} \qquad \frac{44}{1100} = \frac{44}{11} \cdot \frac{1}{100} = 4 \cdot \frac{1}{100} = 0.04.$$

The expression simplifies to

$$\frac{44}{11} + \frac{110}{44} + \frac{44}{1100} = 4 + 2.5 + 0.04 = 6.54.$$

3. Four squares of side length 4, 7, 9, and 10 units are arranged in increasing size order so that their left edges and bottom edges align. The squares alternate in color white-gray-white-gray, respectively, as shown in the figure. What is the area of the visible gray region in square units?



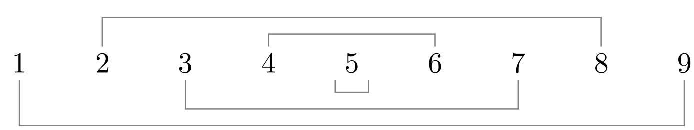
(A) 42 **(B)** 45 **(C)** 49 **(D)** 50 **(E)** 52

Answer (E): The four squares have side lengths of 4, 7, 9, and 10 units. The outer gray region has an area of $10^2 - 9^2 = 100 - 81 = 19$ square units, and the inner gray region has an area of $7^2 - 4^2 = 49 - 16 = 33$ square units. Thus the two gray regions have a combined area of $19 + 33 = 52$ square units.

4. When Yunji added all the integers from 1 through 9, she mistakenly left out a number. Her incorrect sum turned out to be a square number. Which number did Yunji leave out?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (E): Pair up the numbers—1 with 9, 2 with 8, 3 with 7, and 4 with 6—to produce 4 pairs of numbers that sum to 10, then add 5 to find the correct sum of $4 \cdot 10 + 5 = 45$.



The largest square number less than 45 is $6^2 = 36$, which is 9 less than the correct total. Any smaller square number will differ from 45 by more than 9. Therefore 9 is the number Yunji left out.

Note: In general, the sum of the integers from 1 to n equals $\frac{n(n+1)}{2}$. Substituting $n = 9$ gives the sum $\frac{9 \cdot 10}{2} = 9 \cdot 5 = 45$.

5. Aaliyah rolls two standard 6-sided dice. She notices that the product of the two numbers rolled is a multiple of 6. Which of the following integers *cannot* be the sum of the two numbers?

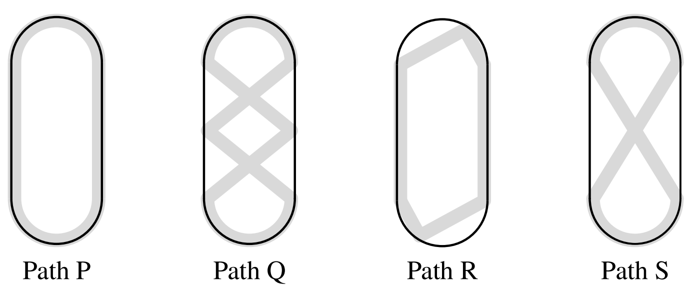
(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Answer (B): Swapping the order of the dice does not change their product or sum, so assume that the first number is less than or equal to the second number. If the product is a multiple of 6, then either one of the numbers is 6, or one number is 3 and the other number is divisible by 2. The table below shows the possible rolls and their corresponding sums and products.

Roll	Sum	Product
$(2,3)$	5	6
$(1,6)$	7	6
$(3,4)$	7	12
$(2,6)$	8	12
$(3,6)$	9	18
$(4,6)$	10	24
$(5,6)$	11	30
$(6,6)$	12	36

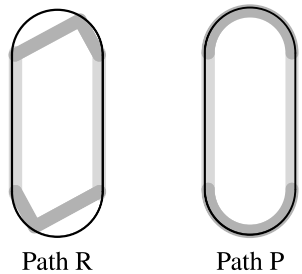
The sums 5 through 12 are all possible with the exception of 6. Therefore 6 cannot be the sum of the two numbers rolled.

6. Sergei skated around an ice rink, gliding along different paths. The gray lines in the figures below show four of the paths labeled P, Q, R, and S. What is the sorted order of the four paths from shortest to longest?

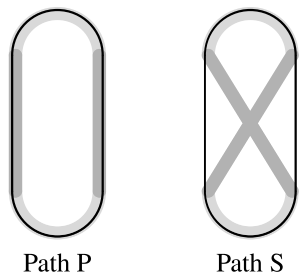


(A) P, Q, R, S **(B)** P, R, S, Q **(C)** Q, S, P, R **(D)** R, P, S, Q **(E)** R, S, P, Q

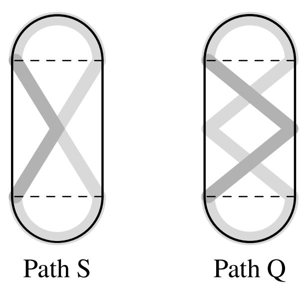
Answer (D): Path R is shorter than Path P because its straight line segments at the top and bottom of the rink are shorter than the corresponding circular arcs of Path P.



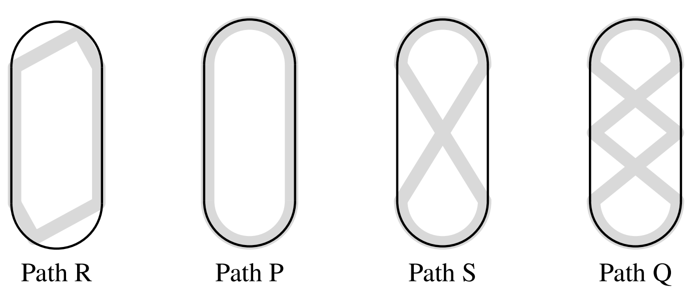
Path P is shorter than Path S because the segments along the sides of the rink are shorter than the diagonals that cut across the rink.



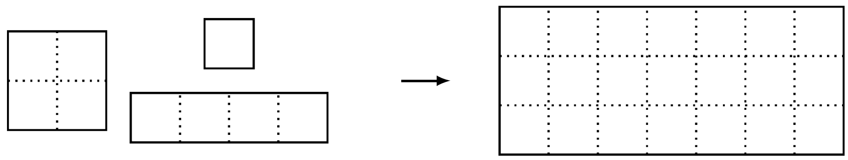
Finally, Path S is shorter than Path Q. This can be seen by ignoring the circular arcs they have in common and focusing on the rectangular region in the center of the rink. Path Q has two overlapping V-shaped lengths in the rectangular region, each length longer than one of the V-shaped lengths in the center of Path S.



In summary, Path R is shorter than path P, which is shorter than path S, which is shorter than Path Q, so the sorted order is R, P, S, Q.



7. A 3×7 rectangle is covered without overlap by 3 shapes of tiles: 2×2, 1×4, and 1×1, shown below. What is the minimum possible number of 1×1 tiles used?

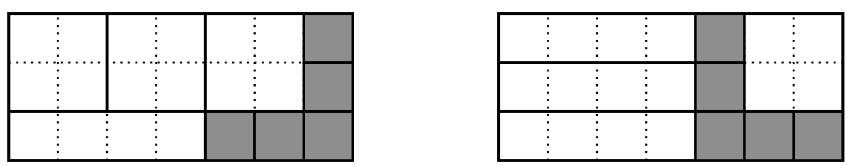


(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer (E): Each 2×2 and 1×4 tile covers an area of 4 square units, so the total area they cover is a multiple of 4. The 3×7 rectangle has an area of 21 square units, which is 1 more than a multiple of 4. Hence the number of 1×1 tiles used must be 1 more than a multiple of 4, for example 1 or 5 or 9 tiles.

First check to see whether a tiling is possible using a single 1×1 tile. Notice that each row of the rectangle has 7 units but each 2×2 and 1×4 tile covers an even number of these units. Thus each of the 3 rows must include at least 1 of the 1×1 tiles, so the tiling must include at least 3 of these tiles.

Next check to see whether a tiling is possible using 5 of the 1×1 tiles. Such a tiling is possible. Two sample tilings are shown below. Therefore the minimum possible number of 1×1 tiles used is 5.



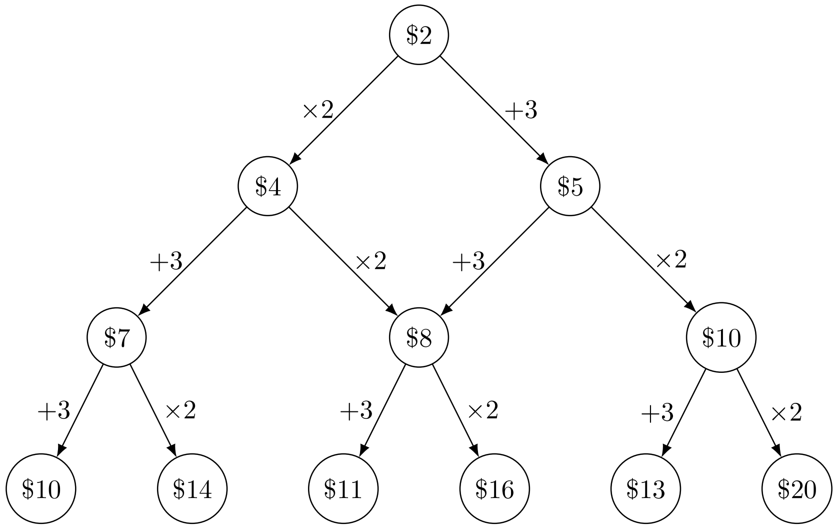
8. On Monday Taye has \$2. Every day, he either gains \$3 or doubles the amount of money he had on the previous day. How many different dollar amounts could Taye have on Thursday, 3 days later?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Answer (D): The following table lists the amounts of money Taye could have on each day. To calculate the possible amounts, add \$3 or double the amount from the previous day, skipping any duplicates.

Day	0 (Monday)	1 (Tuesday)	2 (Wednesday)	3 (Thursday)
Possible amounts	\$2	\$4 = 2 · \$2	\$7 = \$3 + \$4	\$10 = \$3 + \$7
		\$5 = \$3 + \$2	\$8 = \$3 + \$5	\$11 = \$3 + \$8
			\$8 = 2 · \$4	\$13 = \$3 + \$10
			\$10 = 2 · \$5	\$14 = 2 · \$7
				\$16 = 2 · \$8
				\$20 = 2 · \$10

After 3 days, there are 6 different dollar amounts. The graph below shows all the possibilities.



9. All of the marbles in Maria's collection are red, green, or blue. Maria has half as many red marbles as green marbles and twice as many blue marbles as green marbles. Which of the following could be the total number of marbles in Maria's collection?

(A) 24 **(B)** 25 **(C)** 26 **(D)** 27 **(E)** 28

Answer (E): Maria can divide her marbles into piles in such a way that each pile contains exactly 1 red marble, 2 green marbles, and 4 blue marbles. Because each pile contains $1 + 2 + 4 = 7$ marbles, the total number of marbles in her collection must be a multiple of 7. The table below confirms this result for 5 or fewer red marbles. In each case, the total number of marbles is a multiple of 7.

Red	Green	Blue	Total
1	2	4	7
2	4	8	14
3	6	12	21
4	8	16	28
5	10	20	35

Of the given answer choices, only $28 = 7 \cdot 4$ is a multiple of 7.

OR

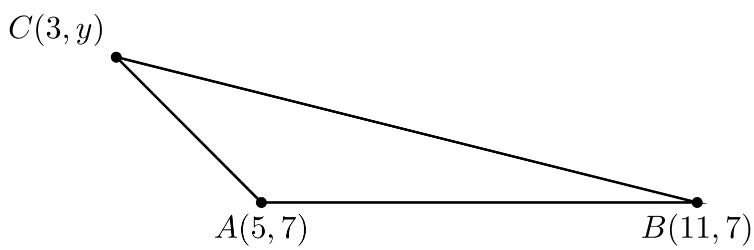
Let the number of red marbles be r. Then the number of green marbles is $2r$ and the number of blue marbles is $2 \cdot 2r = 4r$. The total number of marbles in Maria's collection is $r + 2r + 4r = 7r$, which is a multiple of 7. Of the given answer choices, only $28 = 7 \cdot 4$ is a multiple of 7.

10. In January 1980 the Mauna Loa Observatory recorded carbon dioxide (CO_2) levels of 338 ppm (parts per million). Over the years the average CO_2 reading has increased by about 1.515 ppm each year. What is the expected CO_2 level in ppm in January 2030? Round your answer to the nearest integer.

    (A) 399 **(B)** 414 **(C)** 420 **(D)** 444 **(E)** 459

    Answer (B): Over the 50 years from 1980 to 2030, CO_2 levels are expected to increase at the rate of 1.515 ppm per year, so the total increase is $50 \cdot 1.515 = 75.75$, which rounds to 76 ppm. Therefore the expected CO_2 reading in 2030 is $338 + 76 = 414$ ppm.

11. The coordinates of $\triangle ABC$ are $A(5,7)$, $B(11,7)$ and $C(3,y)$, with $y > 7$. The area of $\triangle ABC$ is 12. What is the value of y ?

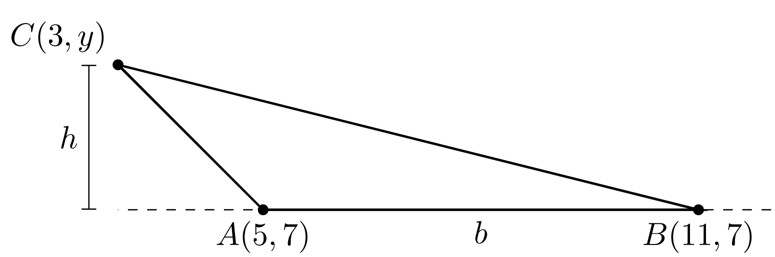


    (A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

    Answer (D): The area K of a triangle with base b and height h is $K = \frac{1}{2}bh$. For $\triangle ABC$, the length of base $\overline{AB}$ is $11 - 5 = 6$. Setting the area K equal to 12 and solving for h gives

$$\begin{aligned} K &= \frac{1}{2}bh \\ 12 &= \frac{1}{2} \cdot 6h \\ 12 &= 3h \\ h &= 4. \end{aligned}$$

    Therefore point $C(3,y)$ is 4 units above the horizontal line through points A and B, so $y = 7 + 4 = 11$. Note that the x-coordinate of point C is irrelevant.



12. Rohan keeps a total of 90 guppies in 4 fish tanks.

   - There is 1 more guppy in the 2nd tank than in the 1st tank.
   - There are 2 more guppies in the 3rd tank than in the 2nd tank.
   - There are 3 more guppies in the 4th tank than in the 3rd tank.

   How many guppies are in the 4th tank?

   (A) 20 **(B)** 21 **(C)** 23 **(D)** 24 **(E)** 26

   Answer (E): The differences in the number of guppies can be satisfied by letting the 1st through 4th tanks contain 0, 1, 3, and 6 guppies, respectively, for a total of 10 guppies. Next evenly distribute the remaining $90 - 10 = 80$ guppies among the 4 tanks. Then the 4th tank will contain 6 guppies plus $\frac{80}{4} = 20$ guppies, for a total of 26 guppies.

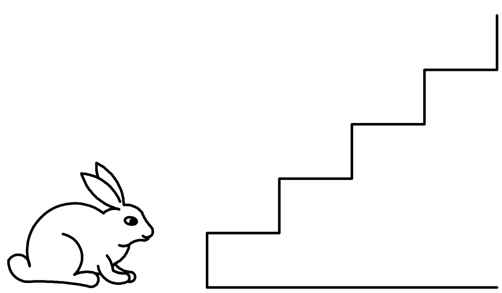
   OR

   Denote the number of guppies in the 1st tank by x. Then the 2nd tank has $x + 1$ guppies, the 3rd tank has $(x + 1) + 2 = x + 3$ guppies, and the 4th tank has $(x + 3) + 3 = x + 6$ guppies. Solving

   $$x + (x + 1) + (x + 3) + (x + 6) = 4x + 10 = 90$$

   gives $x = \frac{90-10}{4} = 20$ guppies in the 1st tank. Then the 4th tank has $x + 6 = 20 + 6 = 26$ guppies.

13. Buzz Bunny is hopping up and down a set of stairs, one step at a time. In how many ways can Buzz start on the ground, make a sequence of 6 hops, and end up back on the ground? (For example, one sequence of hops is up-up-down-down-up-down.)



   (A) 4 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 12

   Answer (B): In order to begin and end on the ground, Buzz must make 3 hops up and 3 hops down in some order. The first hop must move Buzz up off the ground, and the last hop must move Buzz back down to the ground. Let the letters U and D represent up and down hops, respectively. Then the sequence of hops must have the form

   U __ __ __ __ D

with the middle 4 hops consisting of 2 up hops and 2 down hops. Buzz must stay on or above the ground, so at any point in the sequence, the number of down hops cannot be greater than the number of up hops.

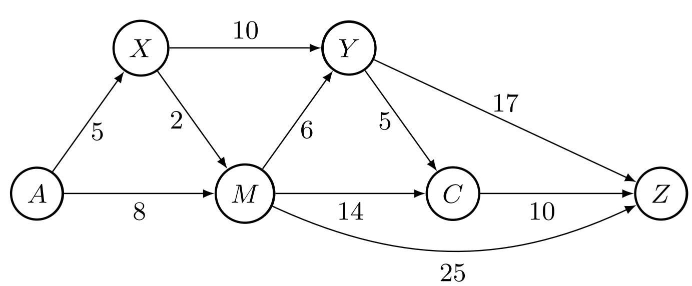
There are 6 ways to arrange the letters UUDD representing the middle 4 hops. One of the arrangements leads to the sequence UDDUUD, which begins with 1 up hop followed by 2 down hops, which would move Buzz below the ground, so that sequence is not possible. The other sequences (UUUDDD, UUDUDD, UUDDUD, UDUUDD, UDUDUD) all keep Buzz on or above the ground. Therefore there are 5 possible 6-hop sequences starting and ending on the ground.

OR

This problem is equivalent to counting the number of ways to draw a "mountain range" with 6 strokes up or down, starting and ending on the ground. There are 5 ways, as shown below.

Note: The Catalan numbers are a sequence of positive integers that appear in many counting problems. This hopping problem illustrates the 3rd Catalan number $C_3 = 5$.

14. The one-way routes connecting towns A, M, C, X, Y, and Z are shown in the figure below (not drawn to scale). The distances in kilometers along each route are marked. Traveling along these routes, what is the shortest distance from A to Z in kilometers?



(A) 28 **(B)** 29 **(C)** 30 **(D)** 31 **(E)** 32

Answer (A): The shortest distance from A to Z can be determined by first calculating the shortest distance from A to each of the other towns.

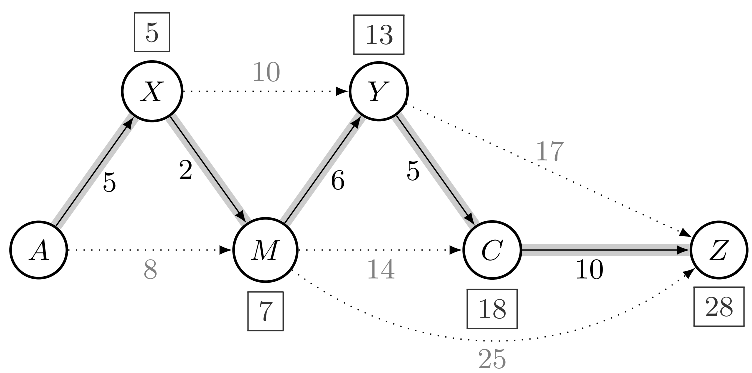
Consider traveling from A to X. There is only one route leading into X, directly from A, so the distance is 5 km.

Next consider traveling from A to M. There are two possible routes. The direct route $A \to M$ has a distance of 8 km, however the indirect route $A \to X \to M$ has a shorter distance of $5 + 2 = 7$ km.

To travel from A to Y, there are two possibilities for the previous town: X (10 km away) or M (6 km away). The shortest distances to X and M have already been determined: 5 and 7 km, respectively. Thus the route to Y via X has a distance of $5 + 10 = 15$ km, and the route via M has a distance of $7 + 6 = 13$ km, which is shorter.

Similarly the distances from A to towns C and Z can be determined by considering the minimum distances to neighboring towns. The shortest routes and distances are shown in the table and figure below.

Destination	Shortest Route	Shortest Distance (km)
X	$A \to X$	5
M	$A \to X \to M$	$5 + 2 = 7$
Y	$A \to X \to M \to Y$	$5 + 2 + 6 = 13$
C	$A \to X \to M \to Y \to C$	$5 + 2 + 6 + 5 = 18$
Z	$A \to X \to M \to Y \to C \to Z$	$5 + 2 + 6 + 5 + 10 = 28$



Therefore the shortest route from A to Z is $A \to X \to M \to Y \to C \to Z$ with a total distance of 28 km.

OR

The road between A and M can be ignored because the shortest route from A to M passes through X. For similar reasons, the roads between X and Y, between M and C, between M and Z, and between Y and Z can be ignored. The only remaining route from A to Z is $A \to X \to M \to Y \to C \to Z$, and its length is $5 + 2 + 6 + 5 + 10 = 28$ kilometers.

Note: This problem is an example of the graph theory *shortest path problem* which finds a path between two vertices (or nodes) that minimizes the sum of the weights along the path. There are many algorithms that can be applied to this problem.

15. Let the letters F, L, Y, B, U, G represent distinct digits. Suppose $\underline{F}\ \underline{L}\ \underline{Y}\ \underline{F}\ \underline{L}\ \underline{Y}$ is the greatest number that satisfies the equation

$$8 \cdot \underline{F}\ \underline{L}\ \underline{Y}\ \underline{F}\ \underline{L}\ \underline{Y} = \underline{B}\ \underline{U}\ \underline{G}\ \underline{B}\ \underline{U}\ \underline{G}\,.$$

What is the value of $\underline{F}\ \underline{L}\ \underline{Y} + \underline{B}\ \underline{U}\ \underline{G}$?

(A) 1089 **(B)** 1098 **(C)** 1107 **(D)** 1116 **(E)** 1125

Answer (C): Because $8 \cdot 125000 = 1000000$ is a seven-digit number, the greatest possible value for $\underline{F}\ \underline{L}\ \underline{Y}\ \underline{F}\ \underline{L}\ \underline{Y}$ is 124124. The product $8 \cdot 124124 = 992992$, however, does not result in distinct digits for $\underline{F}\ \underline{L}\ \underline{Y}$ and $\underline{B}\ \underline{U}\ \underline{G}$. The next number to try is $\underline{F}\ \underline{L}\ \underline{Y}\ \underline{F}\ \underline{L}\ \underline{Y} = 123123$. The product $8 \cdot 123123 = 984984$ does lead to distinct digits for $\underline{F}\ \underline{L}\ \underline{Y} = 123$ and $\underline{B}\ \underline{U}\ \underline{G} = 984$. Therefore the greatest possible value for $\underline{F}\ \underline{L}\ \underline{Y}$ that satisfies the conditions gives a sum of $\underline{F}\ \underline{L}\ \underline{Y} + \underline{B}\ \underline{U}\ \underline{G} = 123 + 984 = 1107$.

OR

The problem can be simplified by observing that $\underline{F}\ \underline{L}\ \underline{Y}\ \underline{F}\ \underline{L}\ \underline{Y}$ equals $1001 \cdot \underline{F}\ \underline{L}\ \underline{Y}$ and $\underline{B}\ \underline{U}\ \underline{G}\ \underline{B}\ \underline{U}\ \underline{G}$ equals $1001 \cdot \underline{B}\ \underline{U}\ \underline{G}$. Then the given equation simplifies to

$$8 \cdot \underline{F}\ \underline{L}\ \underline{Y} = \underline{B}\ \underline{U}\ \underline{G}$$

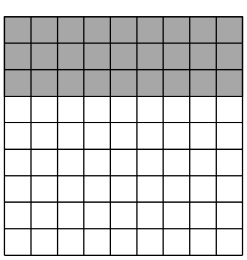
and the solution proceeds as above.

16. Minh enters the numbers 1 through 81 into the cells of a 9×9 grid in some order. She calculates the product of the numbers in each row and column. What is the least number of rows and columns that could have a product divisible by 3?

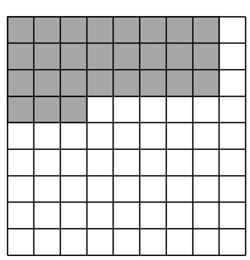
(A) 8 **(B)** 9 **(C)** 10 **(D)** 11 **(E)** 12

Answer (D): Each row or column whose product is divisible by 3 must contain at least one multiple of 3. There are 27 multiples of 3 among the integers from 1 to 81. The number of products divisible by 3 will be minimized if the multiples of 3 are grouped together.

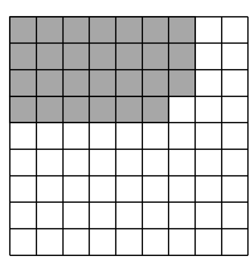
The figures below illustrate the minimum number of products if the 27 multiples span 9, 8, 7, or 6 columns, respectively, resulting in 12 or 11 products. Note that shuffling the rows or columns of the table, or rearranging the multiples of 3, will not change the number of products divisible by 3.



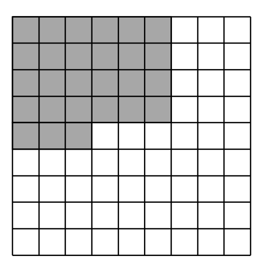
9 columns, 3 rows
$9 + 3 = 12$ products



8 columns, 4 rows
$8 + 4 = 12$ products



7 columns, 4 rows
$7 + 4 = 11$ products



6 columns, 5 rows
$6 + 5 = 11$ products

By symmetry, the calculations for 5, 4, or 3 columns are equivalent to the calculations for 6, 7, or 9 columns, respectively. It is not possible to fit the 27 multiples of 3 into only 1 or 2 columns. Therefore the least number of rows and columns that have a product divisible by 3 is 11.

OR

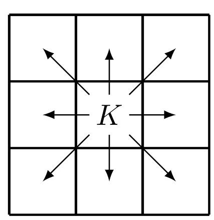
Each row or column whose product is divisible by 3 contains a multiple of 3. Because shuffling the rows and columns of the table does not change the number of rows or columns whose product is divisible by 3, arrange the numbers so that the first x rows and first y columns are the ones containing the multiples of 3.

There are 27 multiples of 3 between 1 and 81 that will fill this x-by-y box. The number of row products and column products divisible by 3 is $x + y$. Thus the original question can be rephrased as: what is the smallest value of $x + y$ given that $xy \geq 27$?

Among all rectangles with a given area, the shape with the least perimeter will be a square. Consider a 5×5 square with $x = 5$ and $y = 5$. Because $xy = 25 < 27$, a sum of $x + y = 10$ will not make a large enough box to fit all the multiples of 3. Thus $x + y \geq 11$. Next consider a 5×6 box with $x = 5$ and $y = 6$. This gives a sum of $x + y = 11$ and a product of $xy = 30 > 27$, which will fit all the multiples of 3. Therefore the least number of rows and columns that have a product divisible by 3 is 11.

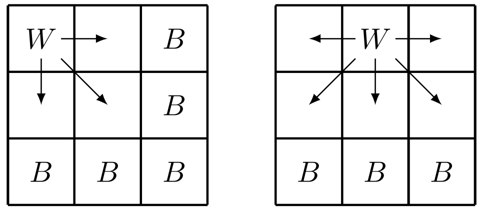
Note that there are other values for (x, y) that will produce the minimum of 11 products, such as $(6, 5)$, $(4, 7)$, and $(7, 4)$.

17. A chess king is said to *attack* all the squares one step away from it, horizontally, vertically, or diagonally. For instance, a king on the center square of a 3×3 grid attacks all 8 other squares, as shown below. Suppose a white king and a black king are placed on different squares of a 3×3 grid so that they do not attack each other. In how many ways can this be done?



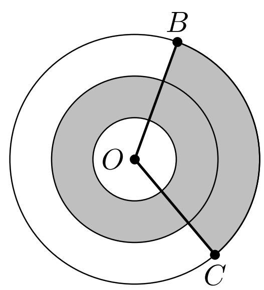
(A) 20 **(B)** 24 **(C)** 27 **(D)** 28 **(E)** 32

Answer (E): Consider the placement of the white king. It cannot occupy the center square, so it must be placed in either a corner square or an edge square. If the white king is on a corner square, there are 5 squares where the black king can be placed. If instead the white king is on an edge square, there are 3 squares where the black king can be placed.



There are 4 corner squares and 4 edge squares, so the total number of ways to place two kings so that they do not attack each other is $4 \cdot 5 + 4 \cdot 3 = 32$.

18. Three concentric circles centered at O have radii of 1, 2, and 3. Points B and C lie on the largest circle. The region between the two smaller circles is shaded, as is the portion of the region between the two larger circles bounded by central angle BOC, as shown in the figure below. Suppose the shaded and unshaded regions are equal in area. What is the measure of $\angle BOC$ in degrees?



(A) 108 **(B)** 120 **(C)** 135 **(D)** 144 **(E)** 150

Answer (A): The three circles have areas of π, 4π, and 9π, from smallest to largest. The area of the shaded middle ring is $4\pi - \pi = 3\pi$, and the area of the entire outer ring is $9\pi - 4\pi = 5\pi$. Because the shaded and unshaded regions are equal in area, half of the total region is shaded, so the area of the shaded region is $\frac{1}{2} \cdot 9\pi = \frac{9}{2}\pi$. Subtracting the area of the shaded middle ring gives an area of $\frac{9}{2}\pi - 3\pi = \frac{3}{2}\pi$ for the shaded portion of the outer ring. Thus the fraction of the outer ring that is shaded is $\frac{3\pi/2}{5\pi} = \frac{3}{10}$, and the measure of $\angle BOC$ is $\frac{3}{10} \cdot 360° = 108°$.

OR

The three circles have areas of π, 4π, and 9π, from smallest to largest. The area of the shaded middle ring is $4\pi - \pi = 3\pi$, and the area of the entire outer ring is $9\pi - 4\pi = 5\pi$. Let x equal the degree measure of $\angle BOC$. Then the shaded portion of the outer ring has area $\frac{x}{360} \cdot 5\pi$ and the unshaded portion of the outer ring has area $5\pi - \frac{x}{360} \cdot 5\pi = \left(1 - \frac{x}{360}\right) \cdot 5\pi$. Setting the total shaded area equal to the total unshaded area gives

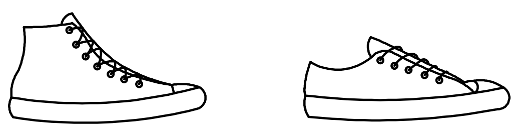
$$3\pi + \frac{x}{360} \cdot 5\pi = \pi + \left(1 - \frac{x}{360}\right) \cdot 5\pi.$$

Solving for x yields

$$\begin{aligned} 2\pi &= \left(1 - \frac{2x}{360}\right) \cdot 5\pi \\ \frac{2}{5} &= 1 - \frac{x}{180} \\ \frac{x}{180} &= \frac{3}{5} \\ x &= \frac{3}{5} \cdot 180 = 3 \cdot 36 = 108. \end{aligned}$$

Note: The region between two concentric circles is called an *annulus*.

19. Jordan owns 15 pairs of sneakers. Three fifths of the pairs are red and the rest are white. Two thirds of the pairs are high-top and the rest are low-top. The red high-top sneakers make up a fraction of the collection. What is the least possible value of this fraction?



(A) 0 **(B)** $\frac{1}{5}$ **(C)** $\frac{4}{15}$ **(D)** $\frac{1}{3}$ **(E)** $\frac{2}{5}$

Answer (C): To minimize the number of red high-top sneakers, maximize the number of red low-top sneakers. Three fifths of the sneakers are red and one third of the sneakers are low-top. If all of the low-top sneakers are red, then the fraction of red high-top sneakers would be $\frac{3}{5} - \frac{1}{3} = \frac{4}{15}$.

OR

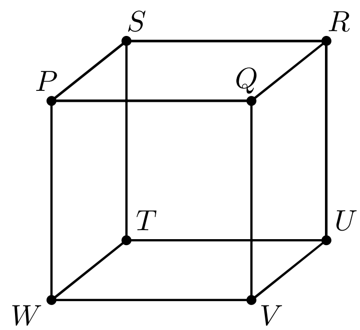
There are 15 pairs of sneakers with $\frac{3}{5} \cdot 15 = 9$ pairs of red sneakers and 6 pairs of white. There are $\frac{2}{3} \cdot 15 = 10$ pairs of high-tops and 5 pairs of low-tops.

	High-top	Low-top	Total
Red	___	___	9
White	___	___	6
Total	10	5	15

The table above can be used to determine the fewest number of red high-tops. Its entry is in the upper left corner. To keep this entry small, the entries in the same row and column should be large. This in turn implies that the number in the lower right corner, corresponding to the white low-tops, should be small. Set this number to 0, then fill in the other entries to find that the smallest possible fraction of red high-top sneakers is $\frac{4}{15}$.

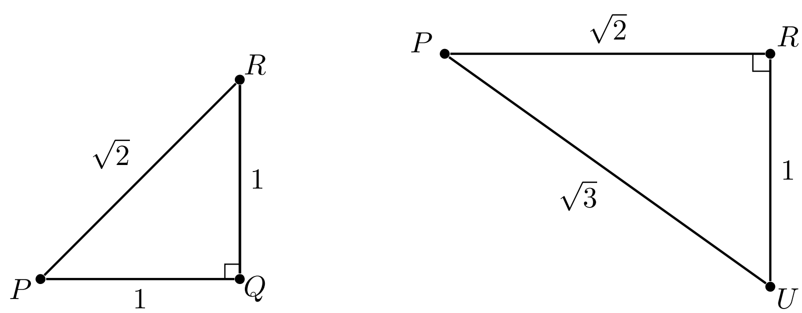
	High-top	Low-top	Total
Red	4	5	9
White	6	0	6
Total	10	5	15

20. Any three vertices of the cube $PQRSTUVW$, shown in the figure below, can be connected to form a triangle. (For example, vertices P, Q, and R can be connected to form isosceles $\triangle PQR$.) How many of these triangles are equilateral and contain P as a vertex?

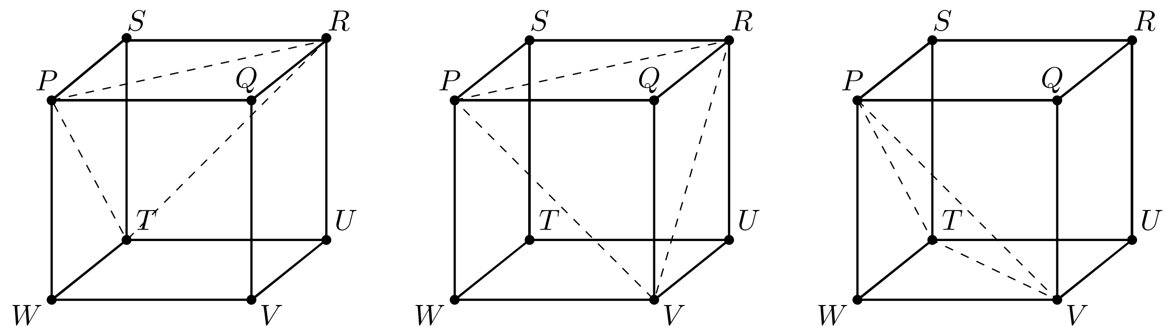


(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 6

Answer (D): Suppose the cube has side length 1. Then the distance between two vertices is 1 if they are endpoints of an edge, $\sqrt{2}$ if they are endpoints of a face diagonal, and $\sqrt{3}$ if they are endpoints of a space diagonal. For example, edge $\overline{PQ}$, face diagonal $\overline{PR}$, and space diagonal $\overline{PU}$ have lengths of 1, $\sqrt{2}$, and $\sqrt{3}$, respectively. These lengths, shown below, can be calculated using the Pythagorean Theorem.



An equilateral triangle cannot be formed from three edges of a cube because edges form 90° angles and an equilateral triangle has 60° angles. No triangle can be formed from three space diagonals because each vertex is an endpoint of exactly one space diagonal. Therefore a triangle can be equilateral only if each of its sides is a face diagonal. There are 3 face diagonals extending from vertex P; they are $\overline{PR}$, $\overline{PT}$, and $\overline{PV}$. Any triangle formed by 2 of these face diagonals will be equilateral because $\overline{RT}$, $\overline{RV}$, and $\overline{TV}$ also are face diagonals. There are 3 ways to select 2 of the 3 face diagonals extending from P. Thus there are 3 equilateral triangles containing P as a vertex, namely $\triangle PRT$, $\triangle PRV$, and $\triangle PTV$.



21. A group of frogs (called an *army*) is living in a tree. A frog turns green when in the shade and turns yellow when in the sun. Initially the ratio of green to yellow frogs was 3 : 1. Then 3 green frogs moved to the sunny side and 5 yellow frogs moved to the shady side. Now the ratio is 4 : 1. What is the difference between the number of green frogs and yellow frogs now?

(A) 10 **(B)** 12 **(C)** 16 **(D)** 20 **(E)** 24

Answer (E): The initial ratio of green to yellow frogs was 3 : 1. This implies that $\frac{3}{3+1} = \frac{3}{4}$ of the frogs were green and $\frac{1}{3+1} = \frac{1}{4}$ of the frogs were yellow. Now the ratio is 4 : 1, so $\frac{4}{4+1} = \frac{4}{5}$ of the frogs are green and $\frac{1}{4+1} = \frac{1}{5}$ are yellow.

Let N represent the total number of frogs in the tree. Before the switch, $\frac{1}{4}$ of the frogs were yellow and now $\frac{1}{5}$ of the frogs are yellow. Thus N must be a multiple of both 4 and 5, so N is a multiple of 20. The table below examines possible values for N: 20, 40, or 60. The number of green and yellow frogs before and after the switch are compared.

	Ratio 3 : 1		Ratio 4 : 1		Change	
N	green	yellow	green	yellow	green	yellow
20	15	5	16	4	+1	−1
40	30	10	32	8	+2	−2
60	45	15	48	12	+3	−3

Because 3 green frogs and 5 yellow frogs switched places, the number of green frogs increased by $5 - 3 = 2$ and the number of yellow frogs decreased by 2. This result matches the result for $N = 40$ frogs. Thus there are now 32 green and 8 yellow frogs with a difference of $32 - 8 = 24$ frogs.

OR

After the switch of positions, the number of yellow frogs decreased by $5 - 3 = 2$, and this decrease represents $\frac{1}{4} - \frac{1}{5} = \frac{1}{20}$ of the total number of frogs. Therefore there are altogether $20 \cdot 2 = 40$ frogs, of which $\frac{1}{5} \cdot 40 = 8$ are yellow and $40 - 8 = 32$ are green. The difference between the number of green and yellow frogs is $32 - 8 = 24$ frogs.

OR

As above, determine that the total number of frogs is 40. Currently $\frac{4}{5}$ of the frogs are green and $\frac{1}{5}$ of the frogs are yellow, which is a difference of $\frac{4}{5} - \frac{1}{5} = \frac{3}{5}$ of the total. Thus the difference between the number of green frogs and yellow frogs is now $\frac{3}{5} \cdot 40 = 24$ frogs.

OR

Denote the initial number of green and yellow frogs by g and y, respectively. The original ratio was $3 : 1$, so the number of green frogs was $g = 3y$. The switch in position increased the number of green frogs by $5 - 3 = 2$, and decreased the number of yellow frogs by 2, resulting in a ratio of $4 : 1$, which leads to the equation

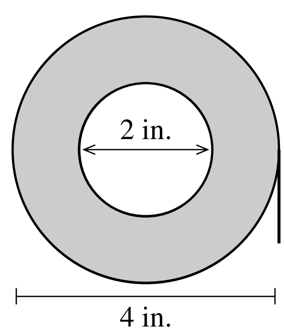
$$\frac{g+2}{y-2} = \frac{4}{1} \quad \Longrightarrow \quad \frac{3y+2}{y-2} = \frac{4}{1}.$$

Cross-multiplying and solving for y gives

$$\begin{aligned} 4(y-2) &= 3y+2 \\ 4y-8 &= 3y+2 \\ y &= 10. \end{aligned}$$

Thus initially there were $y = 10$ yellow frogs and $g = 3 \cdot 10 = 30$ green frogs. Now there are $10 - 2 = 8$ yellow frogs and $30 + 2 = 32$ green frogs. The difference between the number of green and yellow frogs is $32 - 8 = 24$ frogs.

22. A roll of tape is 4 inches in diameter and is wrapped around a ring that is 2 inches in diameter. A cross section of the tape is shown in the figure below. The tape is 0.015 inches thick. If the tape is completely unrolled, approximately how long would it be? Round your answer to the nearest 100 inches.



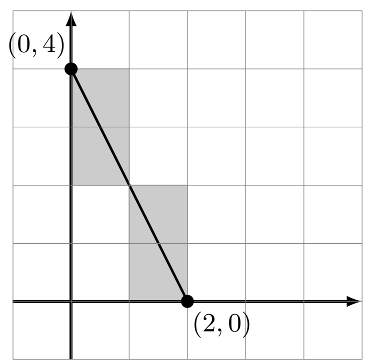
(A) 300 **(B)** 600 **(C)** 1200 **(D)** 1500 **(E)** 1800

Answer (B): A circle of radius r has area πr^2. A cross section of the rolled-up tape has area $\pi\left(2^2 - 1^2\right) = 3\pi$ square inches. A cross section of the unrolled tape is a long rectangle of the same area that is 0.015 inches thick. The length of the unrolled tape equals the area divided by the thickness:

$$\frac{3\pi}{0.015} = \frac{3000\pi}{15} = 200\pi \text{ inches.}$$

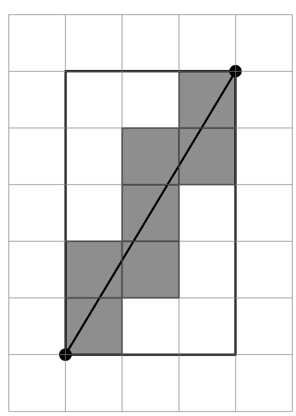
Estimating π as 3.14 gives $200\pi \approx 200(3.14) = 628$ inches. Rounded to the nearest 100 inches, the length of the unrolled tape is approximately 600 inches.

23. Rodrigo has a very large piece of graph paper. First he draws a line segment connecting point $(0, 4)$ to point $(2, 0)$ and colors the 4 cells whose interiors intersect the segment, as shown below. Next Rodrigo draws a line segment connecting point $(2000, 3000)$ to point $(5000, 8000)$. Again he colors the cells whose interiors intersect the segment. How many cells will he color this time?



(A) 6000 **(B)** 6500 **(C)** 7000 **(D)** 7500 **(E)** 8000

Answer (C): The line segment forms the diagonal of a rectangle that is 3000 cells wide and 5000 cells tall. It consists of 1000 shorter segments, each forming the diagonal of a rectangle that is 3 cells wide and 5 cells tall, as pictured below:



Inside each 5-by-3 rectangle, the segment intersects the interiors of 7 cells. Because the line segment from (2000, 3000) to (5000, 8000) consists of 1000 of these segments, Rodrigo will color $7 \cdot 1000 = 7000$ cells.

OR

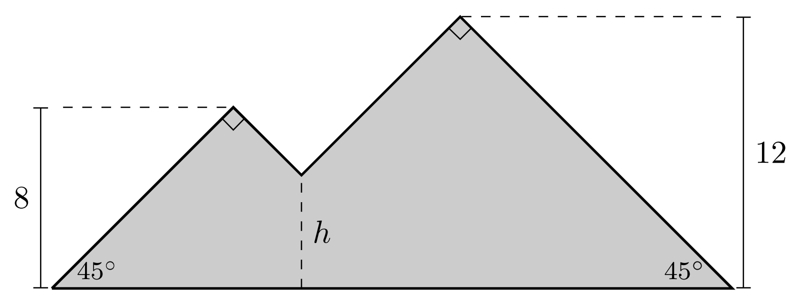
In general, as will be proven below, a diagonal of an $m \times n$ rectangle will pass through the interiors of $m + n - d$ cells where d is the greatest common divisor of m and n.

Consider the case where m and n are relatively prime numbers and $d = \gcd(m, n) = 1$. Observe that a diagonal of an $m \times n$ rectangle will cross $m - 1$ horizontal grid lines and $n - 1$ vertical grid lines. With each crossing, the diagonal enters the interior of a new cell. Adding in the first cell gives a total of $(m - 1) + (n - 1) + 1 = m + n - 1$ cells through which the diagonal will pass. For example, for the 5-by-3 rectangle, the diagonal passes through $5 + 3 - 1 = 7$ cells.

Now consider the case where m and n are not relatively prime. Then as before, a diagonal will meet a grid line $m + n - 1$ times, however there will be $d - 1$ instances where the diagonal intersects both a horizontal and a vertical line at a grid point. When this happens, the diagonal will not enter a new cell. Subtracting these $d - 1$ grid points gives a total of $(m + n - 1) - (d - 1) = m + n - d$ cells through which the diagonal will pass.

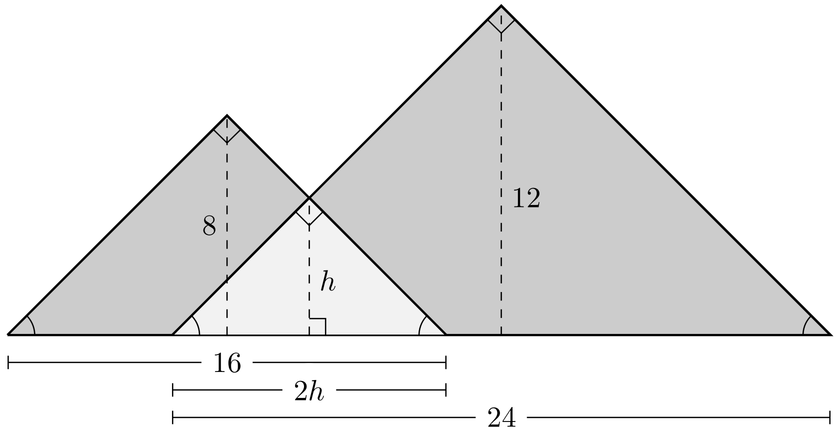
For the given problem, $m = 5000$, $n = 3000$, and their greatest common divisor is $d = 1000$. Therefore Rodrigo will color $m + n - d = 5000 + 3000 - 1000 = 7000$ cells.

24. Jean made a piece of stained glass art in the shape of two mountains, as shown in the figure below. One mountain peak is 8 feet high and the other peak is 12 feet high. Each peak forms a 90° angle, and the straight sides of the mountains form 45° angles with the ground. The artwork has an area of 183 square feet. The sides of the mountains meet at an intersection point near the center of the artwork, h feet above the ground. What is the value of h?



(A) 4 **(B)** 5 **(C)** $4\sqrt{2}$ **(D)** 6 **(E)** $5\sqrt{2}$

Answer (B): Visualize this design as a picture of two mountain peaks surrounding a valley, h feet above the ground. Extend the lines that form the valley down to the ground to form two overlapping isosceles right triangles. The region where they overlap is a third isosceles right triangle of height h.



The area of the entire region can be found by adding the areas of the two larger triangles (with heights 8 and 12) and subtracting the area of the region where they overlap, which is double-counted when adding the areas of the larger triangles. The base of each isosceles right triangle is twice as long as its height. Therefore the total area is

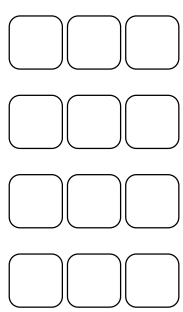
$$A = \frac{1}{2} \cdot 8 \cdot 16 + \frac{1}{2} \cdot 12 \cdot 24 - \frac{1}{2} \cdot h \cdot 2h.$$

Substituting $A = 183$ and solving for h gives

$$\begin{aligned} 183 &= 64 + 144 - h^2 \\ 183 &= 208 - h^2 \\ h^2 &= 25 \\ h &= 5. \end{aligned}$$

The intersection point is 5 feet above the ground.

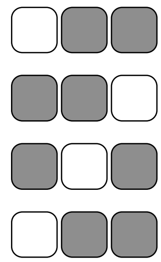
25. A small airplane has 4 rows of seats with 3 seats in each row. Eight passengers have boarded the plane and are distributed randomly among the seats. A married couple is next to board. What is the probability there will be 2 adjacent seats in the same row for the couple?



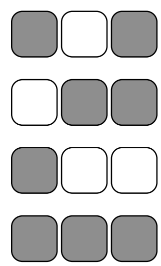
(A) $\frac{8}{15}$ **(B)** $\frac{32}{55}$ **(C)** $\frac{20}{33}$ **(D)** $\frac{34}{55}$ **(E)** $\frac{8}{11}$

Answer (C): There are ${}_{12}C_8 = \frac{12!}{8!4!} = 495$ ways to randomly position the first 8 passengers, resulting in one of four possible distributions across the 4 rows: $(2, 2, 2, 2)$, $(1, 2, 2, 3)$, $(1, 1, 3, 3)$, or $(0, 2, 3, 3)$. Sample arrangements are shown below.

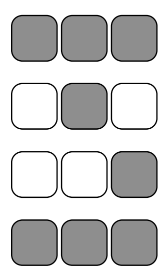
- Case 1: The distribution is $(2, 2, 2, 2)$. With 2 passengers in each row, there will not be adjacent seats for the couple.



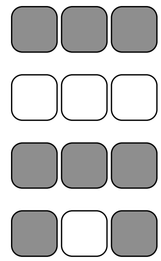
- Case 2: The distribution is $(1, 2, 2, 3)$. There are $\frac{4!}{2!} = 12$ ways to arrange the rows. The row with a single passenger can accommodate the couple if the passenger sits in one of the 2 end seats. Each row with 2 passengers has 3 possible arrangements, so the total number of arrangements is $12 \cdot 2 \cdot 3 \cdot 3 = 216$.



- Case 3: The distribution is $(1, 1, 3, 3)$. There are $\frac{4!}{2!2!} = 6$ ways to arrange the rows. The 2 rows with single passengers have 3 possible seats for each of those passengers, giving a total of $3 \cdot 3 = 9$ possible arrangements. All 9 of those arrangements will have adjacent seats for the couple except for the 1 arrangement with both middle seats filled. Thus the total number of arrangements is $6 \cdot (9 - 1) = 48$.



- Case 4: The distribution is $(0, 2, 3, 3)$. There are $\frac{4!}{2!} = 12$ ways to arrange the rows. The couple is guaranteed to find adjacent seats in the unoccupied row. The row with 2 passengers has 3 possible arrangements, so the total number of arrangements is $12 \cdot 3 = 36$.



Therefore the probability there will be 2 adjacent seats in the same row for the couple is

$$\frac{216+48+36}{495} = \frac{300}{495} = \frac{20}{33}.$$

OR

Use complementary counting to calculate the probability there *will not* be adjacent seats for the couple. There are ${}_{12}C_8 = \frac{12!}{8!4!} = 495$ ways to randomly position the first 8 passengers, resulting in one of four possible distributions across the 4 rows: $(2,2,2,2)$, $(1,2,2,3)$, $(1,1,3,3)$, or $(0,2,3,3)$.

- Case 1: The distribution is $(2,2,2,2)$. There are 3 ways to arrange the 2 passengers in each row, giving a total of $3^4 = 81$ arrangements with no adjacent seats for the couple.
- Case 2: The distribution is $(1,2,2,3)$. There are $\frac{4!}{2!} = 12$ ways to arrange the rows. The row with a single passenger must have that passenger in the middle seat. Each row with 2 passengers has 3 possible arrangements, so the total number of arrangements is $12 \cdot 3 \cdot 3 = 108$.
- Case 3: The distribution is $(1,1,3,3)$. There are $\frac{4!}{2!2!} = 6$ ways to arrange the rows. Each row with a single passenger must have that passenger in the middle seat. The other rows are filled, so the total number of arrangements is 6.
- Case 4: The distribution is $(0,2,3,3)$. The couple is guaranteed to find adjacent seats.

The probability there will not be adjacent seats for the couple is

$$\frac{81+108+6}{495} = \frac{195}{495} = \frac{13}{33}.$$

Therefore the probability there will be adjacent seats for the couple is

$$1 - \frac{13}{33} = \frac{20}{33}.$$

Problems and Solutions contributed by Hannah Alpert, Helen Beylkin, Silva Chang, Steven Davis, Steven Dunbar, Marta Eso, Thomas Hagedorn, Susan Holtzapple, Steven Klee, Rich Morrow, Bryan Nevarez, Isabella Shen, Jeganathan Sriskandarajah, Patrick Vennebush, David Wells, and Jesse Zhang.