MAA AMC

American Mathematics Competitions

Official Solutions

MAA American Mathematics Competitions

37th Annual

AMC 8

Tuesday, January 18, 2022 through Monday, January 24, 2022

These official solutions give at least one solution for each problem on this year's competition and show that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

Questions and comments about this competition should be sent to:

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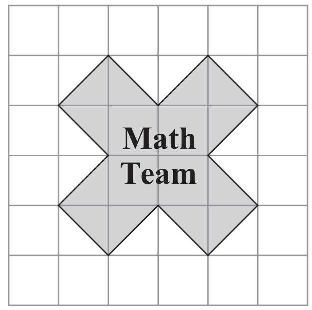
The problems and solutions for this AMC 8 were prepared by the
MAA AMC 8 Editorial Board under the direction of:

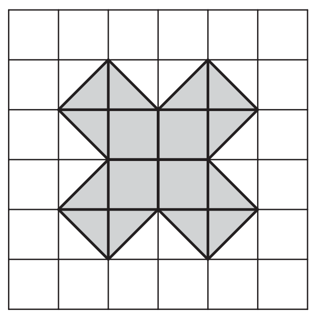
Silva Chang and Zsuzsanna Szaniszlo

1. The Math Team designed a logo shaped like a multiplication symbol, shown below on a grid of 1-inch squares. What is the area of the logo in square inches?



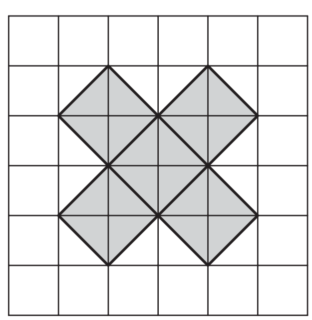
(A) 10 **(B)** 12 **(C)** 13 **(D)** 14 **(E)** 15

Answer (A): The multiplication symbol consists of 4 squares, each measuring 1 inch by 1 inch, surrounded by 12 right triangles, each exactly half of a square. Thus the total area of the logo is $4 \cdot 1 + 12 \cdot \frac{1}{2} = 4 + 6 = 10$ square inches.



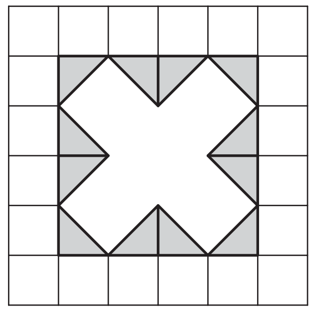
OR

By the Pythagorean Theorem, each 1-by-1-inch square has a diagonal length of $\sqrt{1^2 + 1^2} = \sqrt{2}$. The multiplication symbol consists of 5 congruent squares of side length $\sqrt{2}$. Therefore the total area of the logo is $5\left(\sqrt{2}\right)^2 = 5 \cdot 2 = 10$ square inches.



OR

The logo can be enclosed in a large square of side length 4. There are 12 right triangles, each with area $\frac{1}{2}$ square inch, that are outside the logo and inside that square. The area of the logo equals the area of the large square minus the area of the triangles: $4^2 - 12 \cdot \frac{1}{2} = 16 - 6 = 10$ square inches.



2. Consider these two operations:

$$a \blacklozenge b = a^2 - b^2$$
$$a \bigstar b = (a - b)^2$$

What is the value of $\left(5 \blacklozenge 3\right) \bigstar 6$?

(A) -20 **(B)** 4 **(C)** 16 **(D)** 100 **(E)** 220

Answer (D): The value of the expression in parentheses is

$$5 \blacklozenge 3 = 5^2 - 3^2 = 25 - 9 = 16.$$

Substituting that value into the larger expression gives

$$\left(5 \blacklozenge 3\right) \bigstar 6 = 16 \bigstar 6 = (16 - 6)^2 = 10^2 = 100.$$

3. When three positive integers a, b, and c are multiplied together, their product is 100. Suppose $a < b < c$. In how many ways can the numbers be chosen?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

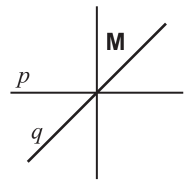
Answer (E): The factors of 100 are 1, 2, 4, 5, 10, 20, 25, 50, and 100. Consider each possible value of a and determine whether there are any solutions given that $a \cdot b \cdot c = 100$ and $a < b < c$.

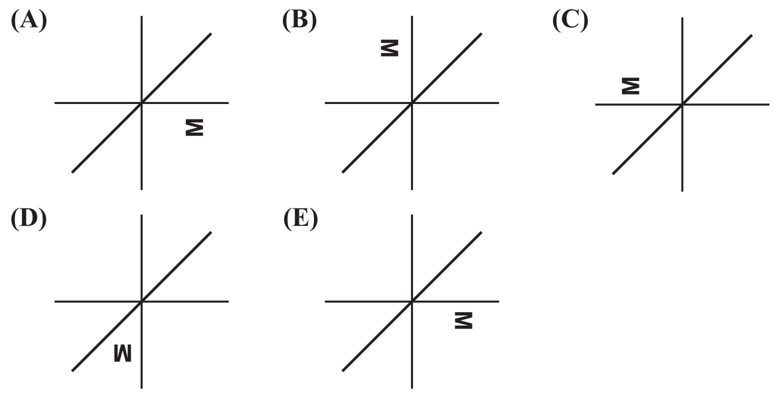
If $a = 1$, then $b \cdot c = 100$. There are 3 solutions: $1 \cdot 2 \cdot 50$, $1 \cdot 4 \cdot 25$, and $1 \cdot 5 \cdot 20$.
If $a = 2$, then $b \cdot c = 50$. There is 1 solution: $2 \cdot 5 \cdot 10$.
If $a \geq 4$, then $b \cdot c \leq 25$. There are no solutions given that $a < b < c$.

In summary, there are 4 ways to choose the numbers a, b, and c:

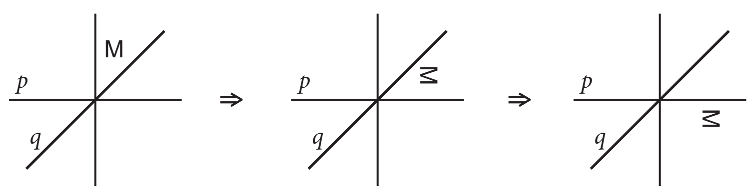
$$100 = 1 \cdot 2 \cdot 50 = 1 \cdot 4 \cdot 25 = 1 \cdot 5 \cdot 20 = 2 \cdot 5 \cdot 10.$$

4. The letter **M** in the figure below is first reflected over the line q and then reflected over the line p. What is the resulting image?





Answer (E): Reflecting the given figure over the line q results in a sideways **M** located below the line q and above the line p, as shown in the middle figure below. Then reflecting that figure over the line p results in the figure in choice **(E)**.



5. Anna and Bella are celebrating their birthdays together. Five years ago, when Bella turned 6 years old, she received a newborn kitten as a birthday present. Today the sum of the ages of the two children and the kitten is 30 years. How many years older than Bella is Anna?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer (C): The kitten is now 5 years old, and Bella is $6 + 5 = 11$ years old. Thus Anna's age is $30 - (5 + 11) = 14$ years, so Anna is $14 - 11 = 3$ years older than Bella.

OR

Let x denote the difference between Anna's and Bella's ages. The table below shows the ages of the children and the kitten 5 years ago and today.

	Bella	Anna	Kitten
5 years ago	6	$6+x$	0
Today	11	$11+x$	5

The sum of their ages today is 30 so $11 + (11 + x) + 5 = 30$, which gives $x = 3$. Thus Anna is 3 years older than Bella.

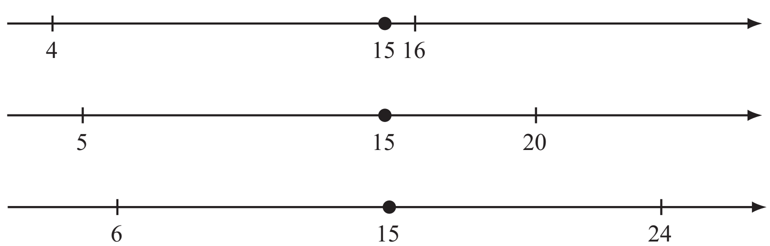
OR

Five years ago the combined ages of Anna, Bella, and the kitten was $30 - 3 \cdot 5 = 15$ years. Because Bella was 6 years old at the time and the kitten was newborn, Anna must have been $15 - 6 = 9$ years old, so Anna is 3 years older than Bella.

6. Three positive integers are equally spaced on a number line. The middle number is 15, and the largest number is 4 times the smallest number. What is the smallest of these three numbers?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Answer (C): Let a and b represent the smallest and largest integers, respectively, with the number 15 between them. The three integers are equally spaced, so a and b must be equidistant from 15. Because b is greater than 15 and is a multiple of 4, the smallest possible value of b is 16 and hence the smallest possible value of a is 4. If $a = 4$ and $b = 16$, the number a will be farther from 15 than b. The same is true if $a = 5$ and $b = 20$. If $a = 6$ and $b = 24$, however, both a and b will be 9 units from 15. For larger values of a, the number a will be closer to 15 than b. Therefore $a = 6$ and $b = 24$ is the only solution, and 6 is the smallest of the three numbers.



OR

Let x and $4x$ represent the smallest and largest numbers, respectively. The distance between the smallest number and the middle number is $15 - x$. The distance between the largest number and the middle number is $4x - 15$. The three integers are equally spaced, so the two distances must be equal. Therefore

$$15 - x = 4x - 15$$
$$30 = 5x$$
$$x = 6.$$

The smallest number is 6.

OR

Let d denote the difference between the first two numbers. The three numbers are equally spaced, so the three numbers can be expressed as $15-d$, 15, and $15+d$. It is given that the largest number is four times the smallest, so

$$15+d=4(15-d).$$

Solving for d gives

$$\begin{aligned}15+d&=60-4d\\5d&=45\\d&=9.\end{aligned}$$

Therefore the smallest number is $15-d=15-9=6$.

7. When the World Wide Web first became popular in the 1990s, download speeds reached a maximum of about 56 kilobits per second. Approximately how many minutes would the download of a 4.2-megabyte song have taken at that speed? (Note that there are 8000 kilobits in a megabyte.)

(A) 0.6 **(B)** 10 **(C)** 1800 **(D)** 7200 **(E)** 36000

Answer (B): There are 8000 kilobits in a megabyte and 60 seconds in a minute. Given a transmission speed of 56 kilobits per second, the approximate download time, in minutes, for a 4.2-megabyte song would have been

$$\frac{4.2\text{ MB}}{56\text{ kbit/sec}}\cdot\frac{8000\text{ kbit}}{1\text{ MB}}\cdot\frac{1\text{ min}}{60\text{ sec}}=10\text{ min}.$$

8. What is the value of

$$\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdots\frac{18}{20}\cdot\frac{19}{21}\cdot\frac{20}{22}?$$

(A) $\frac{1}{462}$ **(B)** $\frac{1}{231}$ **(C)** $\frac{1}{132}$ **(D)** $\frac{2}{213}$ **(E)** $\frac{1}{22}$

Answer (B): Note that all the numbers cancel except for the first two in the numerator and the last two in the denominator. The value of the product is

$$\begin{aligned}\frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdots\frac{18}{20}\cdot\frac{19}{21}\cdot\frac{20}{22}&=\frac{1}{\cancel{3}}\cdot\frac{2}{\cancel{4}}\cdot\frac{\cancel{3}}{\cancel{5}}\cdots\frac{\cancel{18}}{\cancel{20}}\cdot\frac{\cancel{19}}{21}\cdot\frac{\cancel{20}}{22}\\&=\frac{1}{1}\cdot\frac{2}{1}\cdot\frac{1}{21}\cdot\frac{1}{22}=\frac{1}{231}.\end{aligned}$$

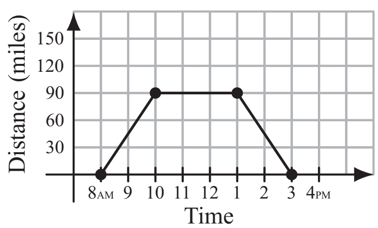
9. A cup of boiling water (212°F) is placed to cool in a room whose temperature remains constant at 68°F. Suppose the difference between the water temperature and the room temperature is halved every 5 minutes. What is the water temperature, in degrees Fahrenheit, after 15 minutes?

   (A) 77 **(B)** 86 **(C)** 92 **(D)** 98 **(E)** 104

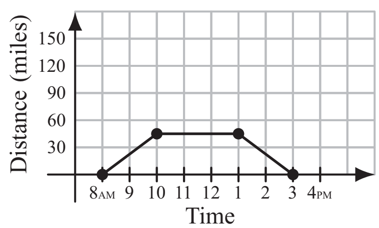
   Answer (B): The initial temperature difference is $212 - 68 = 144$ degrees, so during the first 5 minutes the difference decreases by half to 72 degrees. The next 5 minutes reduces the difference by half again to 36 degrees, and another 5 minutes shrinks the difference to 18 degrees. Thus after 15 minutes the water temperature has dropped to $68 + 18 = 86$ degrees Fahrenheit.

10. One sunny day, Ling decided to take a hike in the mountains. She left her house at 8 AM, drove at a constant speed of 45 miles per hour, and arrived at the hiking trail at 10 AM. After hiking for 3 hours, Ling drove home at a constant speed of 60 miles per hour. Which of the following graphs best illustrates the distance between Ling's car and her house over the course of her trip?

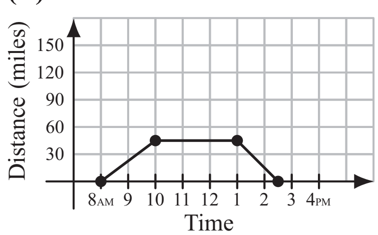
(A)



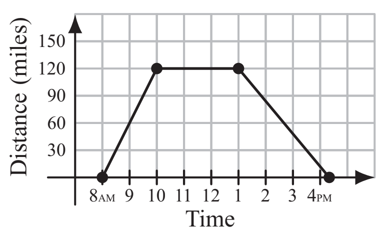
(B)



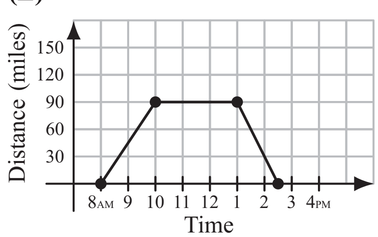
(C)



(D)



(E)



Answer (E): It took Ling 2 hours to drive from her house to the hiking trail, driving at a speed of 45 miles per hour. This means the trail was $2 \cdot 45 = 90$ miles from her house. She arrived at 10 AM and hiked for 3 hours, so she left the trail at 1 PM. She then drove home, covering a distance of 90 miles at a speed of 60 miles per hour. Hence the return trip took $90 \div 60 = 1.5$ hours, and Ling arrived at home at 2:30 PM.

Graphs (A) and (E) are the only ones where Ling drove 90 miles to reach the trail. Graphs (C) and (E) are the only ones where she returns home at 2:30 PM. This means the correct answer must be **(E)**.

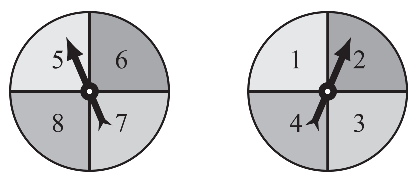
11. Henry the donkey has a very long piece of pasta. He takes a number of bites of pasta, each time eating 3 inches of pasta from the middle of one piece. In the end, he has 10 pieces of pasta whose total length is 17 inches. How long, in inches, was the piece of pasta he started with?

(A) 34 **(B)** 38 **(C)** 41 **(D)** 44 **(E)** 47

Answer (D): Each bite increases the number of pieces of pasta by 1, so to end with 10 pieces of pasta, Henry must have taken 9 bites. Those 9 bites led him to eat $9 \cdot 3 = 27$ inches of pasta. There were 17 inches of pasta remaining at the end, so the length of the piece he started with was $27 + 17 = 44$ inches.

12. The arrows on the two spinners shown below are spun. Let the number N equal 10 times the number on Spinner A, added to the number on Spinner B. What is the probability that N is a perfect square number?

(A) $\frac{1}{16}$ **(B)** $\frac{1}{8}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{3}{8}$ **(E)** $\frac{1}{2}$



Answer (B): The smallest possible value of N is $5 \cdot 10 + 1 = 51$ and the largest possible value is $8 \cdot 10 + 4 = 84$. The only perfect squares in this range are $8^2 = 64$ and $9^2 = 81$, both of which are possible values of N. There are $4 \cdot 4 = 16$ distinct numbers that can be formed from the spinners, and each number is equally likely. The probability that the number N is a perfect square is therefore $\frac{2}{16} = \frac{1}{8}$.

13. How many positive integers can fill the blank in the sentence below?

"One positive integer is ____ more than twice another, and the sum of the two numbers is 28."

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Answer (D): Let n be the missing number, and let a and b be the two numbers that sum to 28, with $a < b$. Because b is more than twice a, the value of a cannot exceed $\frac{1}{3}$ of 28. It follows that the possible values of a are the integers from 1 to 9. The table below shows the corresponding values of b, which equals the difference between 28 and a, and the values of n, which equals the difference between b and $2a$.

a	1	2	3	4	5	6	7	8	9
b	27	26	25	24	23	22	21	20	19
n	25	22	19	16	13	10	7	4	1

Each value of a produces a different value of n, so there are 9 values of n that can fill the blank in the given sentence.

OR

Let n be the missing number, and let a be the smaller of the two numbers in the problem. Then the larger number is $2a + n$, and their sum is

$$28 = a + (2a + n) = 3a + n.$$

Both a and n are positive integers so the possible values of a are the integers from 1 to 9, each corresponding to a distinct positive integer n. Thus there are 9 values of n that can fill the blank in the given sentence.

14. In how many ways can the letters in BEEKEEPER be rearranged so that two or more Es do not appear together?

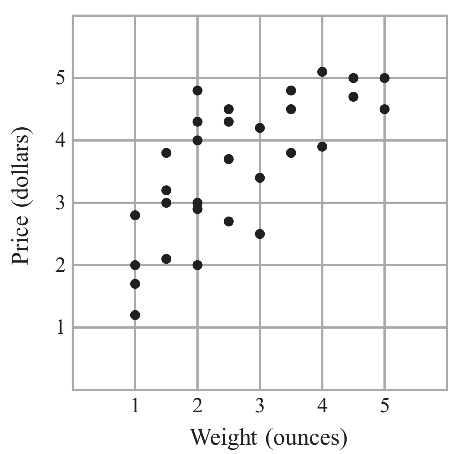
(A) 1 **(B)** 4 **(C)** 12 **(D)** 24 **(E)** 120

Answer (D): The word BEEKEEPER has 9 letters, 5 of which are Es. This means the Es must alternate, starting from the first letter:

$$\underline{E}\ \_\ \underline{E}\ \_\ \underline{E}\ \_\ \underline{E}\ \_\ \underline{E}$$

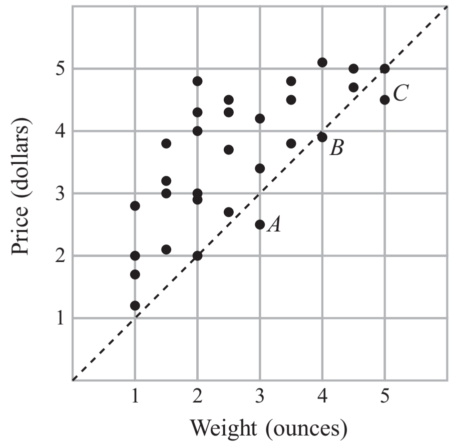
The remaining 4 letters, B, K, P, and R, must be placed in the blank spaces. There are 4 possible letters that can go in the first space, leaving 3 remaining letters to go in the second space, 2 for the third space, and 1 for the last space. Thus there are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ possible arrangements.

15. Laszlo went online to shop for black pepper and found thirty different black pepper options varying in weight and price, shown in the scatter plot below. In ounces, what is the weight of the pepper that offers the lowest price per ounce?



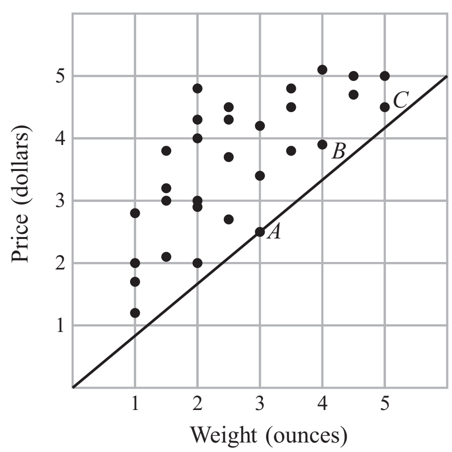
(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer (C):



The dashed line in the figure above passes through the peppers that cost exactly \$1 per ounce. The peppers above the line are more expensive. The only peppers that cost less than \$1 per ounce are the three peppers below the dashed line, labeled A, B, and C. Pepper A costs about \$2.50 for 3 ounces, which is \$0.83 per ounce; pepper B costs a little less than \$1 per ounce; and pepper C costs about \$4.50 for 5 ounces, which is \$0.90 per ounce. Thus the lowest price per ounce is offered by pepper A which weighs 3 ounces.

OR



If a line is drawn from the origin to any of the peppers, the slope of the line corresponds to the price per ounce for that pepper. The line with the smallest slope, as shown above, is the one connecting the origin to pepper A, which weighs 3 ounces.

16. Four numbers are written in a row. The average of the first two is 21, the average of the middle two is 26, and the average of the last two is 30. What is the average of the first and last of the numbers?

(A) 24 **(B)** 25 **(C)** 26 **(D)** 27 **(E)** 28

Answer (B): The sum of the first two numbers is $2 \cdot 21 = 42$ and the sum of the last two is $2 \cdot 30 = 60$, so the sum of all four is $42 + 60 = 102$. The sum of the middle two is $2 \cdot 26 = 52$, so the sum of the first and last is $102 - 52 = 50$, and their average is $\frac{50}{2} = 25$.

OR

Let the four numbers be a, b, c, and d. It is given that

$$\frac{a+b}{2} = 21, \quad \frac{b+c}{2} = 26, \quad \text{and} \quad \frac{c+d}{2} = 30,$$

which implies that $a + b = 42$, $b + c = 52$, and $c + d = 60$. Adding the three equations yields $a + 2(b + c) + d = 154$, and substituting $b + c = 52$ produces $a + d = 154 - 2 \cdot 52 = 50$. Therefore the average of the first and last numbers is

$$\frac{a+d}{2} = \frac{50}{2} = 25.$$

17. If n is an even positive integer, the *double factorial* notation $n!!$ represents the product of all the even integers from 2 to n. For example, $8!! = 2 \cdot 4 \cdot 6 \cdot 8$. What is the units digit of the following sum?

$$2!! + 4!! + 6!! + \cdots + 2018!! + 2020!! + 2022!!$$

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

Answer (B):

The values of $n!!$ for the first few terms in the sum are shown in the table below:

n	$n!!$	Units Digit
2	2	2
4	$2 \cdot 4$	8
6	$2 \cdot 4 \cdot 6$	8
8	$2 \cdot 4 \cdot 6 \cdot 8$	4
10	$2 \cdot 4 \cdot 6 \cdot 8 \cdot 10$	0
12	$2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12$	0

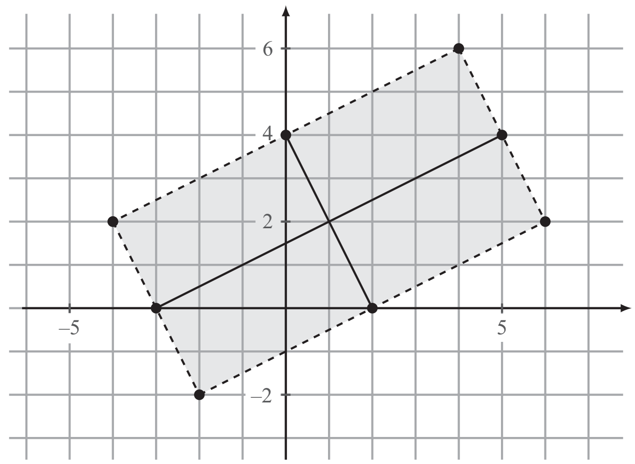
The last two rows of the table show that for even numbers $n \geq 10$, the number 10 will appear as one of the factors in $n!!$, and hence the units digit of $n!!$ will be 0. Thus the units digit of the sum given in the problem is the same as the units digit of $2 + 8 + 8 + 4 = 22$, so the answer is 2.

Note: If n is an odd positive integer, then $n!!$ represents the product of all the odd integers from 1 to n. For example, $9!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9$.

18. The midpoints of the four sides of a rectangle are $(-3,0)$, $(2,0)$, $(5,4)$, and $(0,4)$. What is the area of the rectangle?

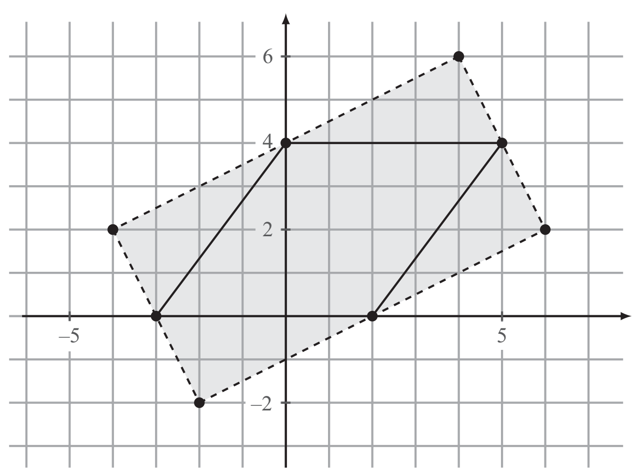
(A) 20 **(B)** 25 **(C)** 40 **(D)** 50 **(E)** 80

Answer (C):



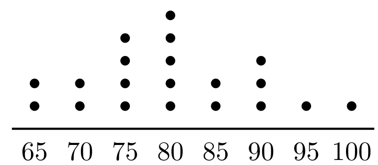
The width w of the rectangle corresponds to the distance between opposite midpoints $(-3,0)$ and $(5,4)$. By the Pythagorean Theorem, that distance equals $\sqrt{(5-(-3))^2+4^2} = \sqrt{8^2+4^2} = \sqrt{80}$. Similarly the height h of the rectangle corresponds to the distance between midpoints $(2,0)$ and $(0,4)$, which equals $\sqrt{2^2+4^2} = \sqrt{20}$. The area of the rectangle is therefore $w \cdot h = \sqrt{80} \cdot \sqrt{20} = \sqrt{1600} = 40$.

OR



The four midpoints form a rhombus with area equal to half the area of the rectangle. The rhombus is a parallelogram with base 5 and height 4, so its area is $5 \cdot 4 = 20$. Thus the area of the rectangle is $2 \cdot 20 = 40$.

19. Mr. Ramos gave a test to his class of 20 students. The dot plot below shows the distribution of test scores.



Later Mr. Ramos discovered that there was a scoring error on one of the questions. He regraded the tests, awarding some of the students 5 extra points, which increased the median test score to 85. What is the minimum number of students who received extra points?

(Note that the *median* test score equals the average of the 2 scores in the middle if the 20 test scores are arranged in increasing order.)

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Answer (C): If the 20 scores are arranged in increasing order, the median will equal the average of the 2 numbers in the 10th and 11th positions. The dot plot shows that after the initial round of grading, the middle 8 scores were

$$\ldots\ 75\ 75\ 80\ 80\ \Big|\ 80\ 80\ 80\ 85\ \ldots,$$

so the median was 80. After regrading, the median increased to 85, so the scores in the 10th and 11th positions must have increased by 5 points, along with the 2 scores of 80 in the next 2 positions:

$$\ldots\ 75\ 75\ 80\ 85\ \Big|\ 85\ 85\ 85\ 85\ \ldots.$$

Other scores might have increased as well but those changes would not have affected the median. Therefore a minimum of 4 students received extra points.

OR

After the first round of grading, there were 7 scores of 85 or greater. As a result of the regrade, the median increased to 85, so the middle 2 numbers must have been 85 and the 9 scores to the right must have been the same or greater. Therefore a minimum of $11-7=4$ students received extra points.

20. The grid below is to be filled with integers in such a way that the sum of the numbers in each row and the sum of the numbers in each column are the same. Four numbers are missing. The number x in the lower left corner is larger than the other three missing numbers. What is the smallest possible value of x?

-2	9	5
		-1
x		8

(A) -1 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 9

Answer (D): Let y, z, and w be the other missing numbers, as shown in the figure below.

-2	9	5
y	z	-1
x	w	8

The sum of the numbers in each row and column is 12. It follows that $x + y = 14$, $y + z = 13$, $z + w = 3$, and $x + w = 4$. It is given that x is larger then the other missing numbers. The sum $x + y$ equals 14, so $x \geq 8$. If x equals 8, then the values $y = 6$, $z = 7$, and $w = -4$ satisfy the requirements of the problem. Therefore $x = 8$ is the smallest integer that can be placed in the lower left corner.

OR

Let x be the number in the lower left corner. The sum of the numbers in each row and column is 12, so the remaining entries can be expressed in terms of x, as shown below.

-2	9	5
$14 - x$	$x - 1$	-1
x	$4 - x$	8

Note that it is always the case that $x > x - 1$ and $14 - x > 4 - x$. This means that the only condition that must be satisfied is $x > 14 - x$, which implies that $2x > 14$, and hence $x > 7$. Therefore $x = 8$ is the smallest integer that can be placed in the lower left corner.

21. Steph scored 15 baskets out of 20 attempts in the first half of a game, and 10 baskets out of 10 attempts in the second half. Candace took 12 attempts in the first half and 18 attempts in the second. In each half, Steph scored a higher percentage of baskets than Candace. Surprisingly they ended with the same overall percentage of baskets scored. How many more baskets did Candace score in the second half than in the first?

	First Half	Second Half
Steph	$\frac{15}{20}$	$\frac{10}{10}$
Candace	$\frac{\square}{12}$	$\frac{\square}{18}$

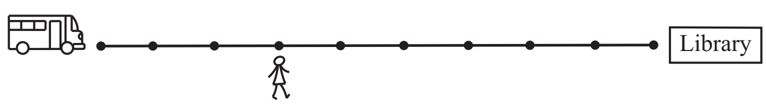
(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Answer (C): Steph and Candace each had a total of 30 attempts. Steph scored a total of $15+10=25$ baskets and the two players ended with the same percentage of baskets scored, so Candace also scored 25 baskets. In the first half of the game, Steph was successful in $\frac{15}{20}=\frac{3}{4}$ of his attempts. Candace was less successful, so she scored fewer than $\frac{3}{4}$ of her 12 attempts, making fewer than 9 baskets. In the second half, Steph succeeded in all of his attempts, but Candace missed at least one of her shots, so she scored fewer than 18 baskets. Because Candace ended with a total of 25 baskets, she must have scored 8 and 17 baskets in the two halves, respectively, making 9 more baskets in the second half.

	First Half	Second Half
Steph	$\frac{15}{20}=\frac{3}{4}=75\%$	$\frac{10}{10}=100\%$
Candace	$\frac{8}{12}=\frac{2}{3}<75\%$	$\frac{17}{18}<100\%$

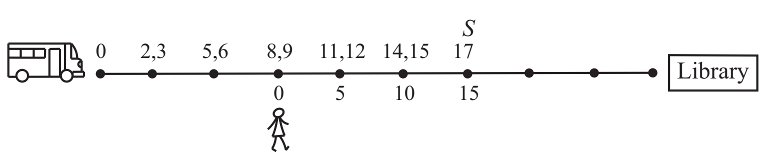
Note: This result is an example of Simpson's paradox. Although one player had a higher percentage score in each half, it was possible for the two players to end with the same overall percentage.

22. A bus takes 2 minutes to drive from one stop to the next, and waits 1 minute at each stop to let passengers board. Zia takes 5 minutes to walk from one bus stop to the next. As Zia reaches a bus stop, if the bus is at the previous stop or has already left the previous stop, then she will wait for the bus. Otherwise she will start walking toward the next stop. Suppose the bus and Zia start at the same time toward the library, with the bus 3 stops behind. After how many minutes will Zia board the bus?



(A) 17 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 23

Answer (A): Refer to the figure below. The numbers above the line show the time in minutes when the bus arrives at and leaves each stop. The numbers below the line show the time in minutes when Zia reaches each stop.

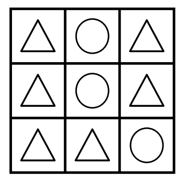


When Zia arrives at bus stop S after 15 minutes, she notices that the bus is just leaving the previous stop, so she will wait for the bus. She will board the bus as soon as it arrives at S, after 17 minutes.

OR

The bus and Zia begin 3 stops apart, with the bus traveling 1 stop in 3 minutes and Zia traveling 1 stop in 5 minutes. She will wait for the bus when it is within 1 stop from her, so the bus needs to make up a distance of 2 stops. In the time Zia travels 1 stop in 5 minutes, the bus travels $\frac{5}{3} = 1\frac{2}{3}$ stops, reducing the distance between them by $\frac{2}{3}$ stop. It will take $\frac{2}{2/3} = 3$ of Zia's stops before the bus is close enough for her to wait. That will happen after $3 \cdot 5 = 15$ minutes when she is at her third stop and the bus is leaving the previous stop. Zia will wait 2 more minutes for the bus to arrive. She will board the bus after a total of $15 + 2 = 17$ minutes.

23. A $\triangle$ or $\bigcirc$ is placed in each of the nine squares in a 3-by-3 grid. Shown below is a sample configuration with three $\triangle$s in a line.



How many configurations will have three $\triangle$s in a line and three $\bigcirc$s in a line?

(A) 39 **(B)** 42 **(C)** 78 **(D)** 84 **(E)** 96

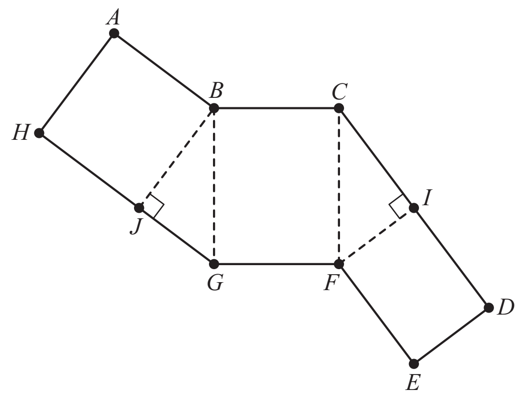
Answer (D): The $\triangle\triangle\triangle$ and $\bigcirc\bigcirc\bigcirc$ lines must have the same orientation: they are either in rows or in columns. First assume that the lines are in rows; the case for columns gives the same number of configurations. There are two cases to consider: either there is exactly one row of each type, or there are two rows of one type and one row of the other.

Case 1: There is exactly one $\triangle\triangle\triangle$ row and one $\bigcirc\bigcirc\bigcirc$ row. There are $3 \cdot 2 = 6$ ways to select the positions of the two rows. For the remaining row, there are $2^3 = 8$ total arrangements, however the symbols cannot be all $\triangle$s or all $\bigcirc$s, leaving 6 possible arrangements. The number of configurations is therefore $6 \cdot 6 = 36$.

Case 2: There are two rows of one type and one row of the other. There are 3 ways to choose the positions of the identical rows and 2 ways to choose the symbol for them. The third row is then determined. The number of configurations is $3 \cdot 2 = 6$.

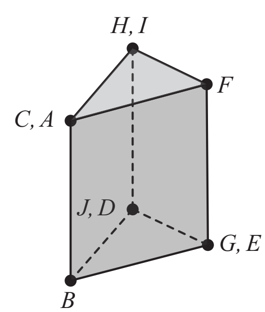
In total there are $36 + 6 = 42$ configurations with $\triangle\triangle\triangle$ and $\bigcirc\bigcirc\bigcirc$ in rows. Doubling to account for the column orientation gives a total of $2 \cdot 42 = 84$ configurations that have three $\triangle$s in a line and three $\bigcirc$s in a line.

24. The figure below shows a polygon $ABCDEFGH$, consisting of rectangles and right triangles. When cut out and folded on the dotted lines, the polygon forms a triangular prism. Suppose that $AH = EF = 8$ and $GH = 14$. What is the volume of the prism?

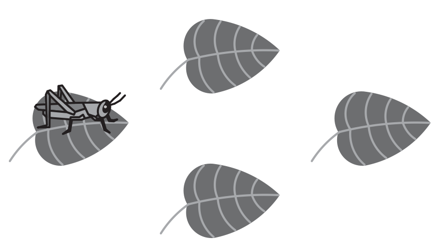


(A) 112 **(B)** 128 **(C)** 192 **(D)** 240 **(E)** 288

Answer (C): When folded, point A coincides with C, point E with G, point H with I, and point D with J. The triangular prism can be oriented to have $\triangle BJG$ as the base and $BC = FG = HJ = DI = 8$ as the height. Note that $GJ = GH - HJ = 14 - 8 = 6$, so BJG is a right triangle with legs of lengths 8 and 6, and an area of $\frac{1}{2} \cdot 8 \cdot 6 = 24$. The volume of a prism is the area of its base multiplied by its height, and so the volume is $24 \cdot 8 = 192$.



25. A cricket randomly hops between 4 leaves, on each turn hopping to one of the other 3 leaves with equal probability. After 4 hops, what is the probability that the cricket has returned to the leaf where it started?



(A) $\frac{2}{9}$ **(B)** $\frac{19}{80}$ **(C)** $\frac{20}{81}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{7}{27}$

Answer (E): Suppose the cricket starts at leaf S. Consider the two 4-hop sequence patterns that start and end at S, shown below, with each blank representing a non-S leaf. For each pattern, the number of sequences is listed. Note that there are 3 ways to move from S to a non-S leaf, and there are 2 ways to move from one non-S leaf to another non-S leaf.

Pattern	Number of Sequences
S __ __ __ S	$3 \cdot 2 \cdot 2 = 12$
S __ S __ S	$3 \cdot 3 = 9$

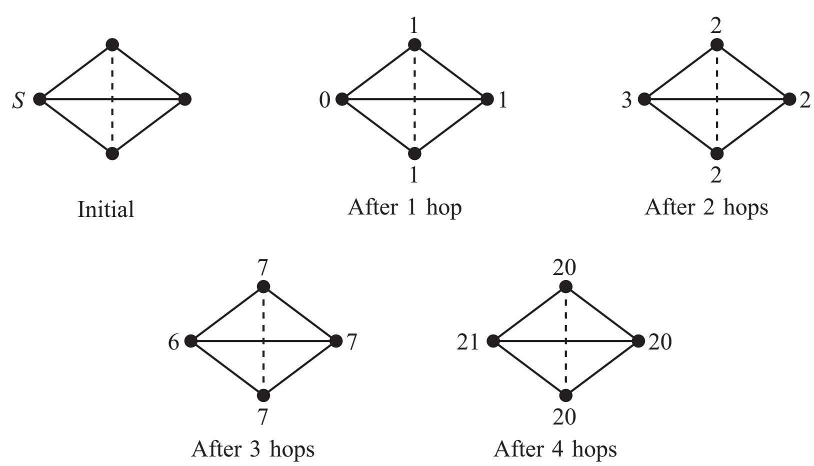
In all there are $3^4 = 81$ possible 4-hop sequences starting at S. The probability that the cricket is back at S after 4 hops is $\frac{12+9}{81} = \frac{21}{81} = \frac{7}{27}$.

OR

Suppose the cricket starts at leaf S. Consider its position after 3 hops and how it can return to S on the 4th hop. There are $3^3 = 27$ possible sequences of 3 hops, and the only sequences that end at S are those that visit two of the other leaves. There are 3 choices for the first leaf and 2 choices for the second leaf, so after 3 hops, the cricket is at S with probability $\frac{3 \cdot 2}{27} = \frac{2}{9}$, which implies that it is not at S with probability $1 - \frac{2}{9} = \frac{7}{9}$. On the 4th hop the cricket cannot go from S to S, but it goes from any other leaf to S with probability $\frac{1}{3}$. Therefore the cricket returns to S after 4 hops with probability $\frac{7}{9} \cdot \frac{1}{3} = \frac{7}{27}$.

OR

Suppose the cricket starts at leaf S. The number of ways to reach each leaf after each hop can be found by adding the counts at adjacent leaves.



For example, there are 6 ways to reach leaf S in 3 hops because there are 2 ways to reach each of the non-S leaves in 2 hops. In all there are $21 + 3 \cdot 20 = 81$ possible sequences of 4 hops. The probability that the cricket is back at S after 4 hops is $\frac{21}{81} = \frac{7}{27}$.

OR

A recurrence relation can be used to solve this problem. Let p_n and q_n represent the probability that after n hops the cricket is at S or not at S, respectively, with $p_0 = 1$ and $q_0 = 0$. Then for $n \geq 1$, $p_n = \frac{1}{3}q_{n-1}$ and $q_n = 1 - p_n$. Applying this recurrence leads to the desired probability $p_4 = \frac{7}{27}$.

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