MAA AMC

American Mathematics Competitions

Official Solutions

MAA American Mathematics Competitions

36th Annual

AMC 8

Tuesday, November 10, 2020 through Monday, November 16, 2020

These official solutions give at least one solution for each problem on this year's competition and show that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

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The problems and solutions for this AMC 8 were prepared by the
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1. Luka is making lemonade to sell at a school fundraiser. His recipe requires 4 times as much water as sugar and twice as much sugar as lemon juice. He uses 3 cups of lemon juice. How many cups of water does he need?

(A) 6 **(B)** 8 **(C)** 12 **(D)** 18 **(E)** 24

Answer (E):

The recipe requires twice as much sugar as lemon juice, so Luka needs $2 \cdot 3 = 6$ cups of sugar. The recipe requires 4 times as much water as sugar, so Luka needs $4 \cdot 6 = 24$ cups of water.

OR

The recipe requires 4 times as much water as sugar and twice as much sugar as lemon juice. This means that Luka needs $4 \cdot 2 = 8$ times as much water as lemon juice. The amount of water he needs is $8 \cdot 3 = 24$ cups.

2. Four friends do yardwork for their neighbors over the weekend, earning \$15, \$20, \$25, and \$40, respectively. They decide to split their earnings equally among themselves. In total how much will the friend who earned \$40 give to the others?

(A) \$5 **(B)** \$10 **(C)** \$15 **(D)** \$20 **(E)** \$25

Answer (C):

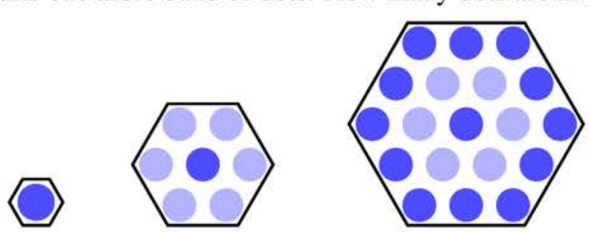
Altogether the four friends earned \$100, so they each will have \$25 after they divide their earnings equally. The friend who earned \$40 will give to the others \$40 − \$25 = \$15.

3. Carrie has a rectangular garden that measures 6 feet by 8 feet. She plants the entire garden with strawberry plants. Carrie is able to plant 4 strawberry plants per square foot, and she harvests an average of 10 strawberries per plant. How many strawberries can she expect to harvest?

(A) 560 **(B)** 960 **(C)** 1120 **(D)** 1920 **(E)** 3840

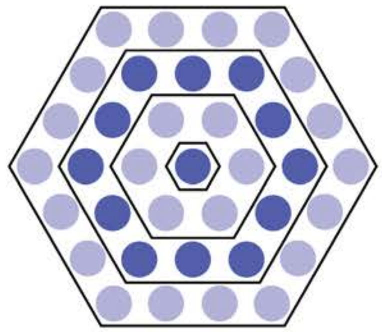
Answer (D): The area of Carrie's garden is $6 \cdot 8 = 48$ square feet. She can plant 4 plants per square foot, so Carrie is able to plant $4 \cdot 48 = 192$ strawberry plants. She can expect to harvest 10 strawberries per plant for a total of $192 \cdot 10 = 1920$ strawberries.

4. Three hexagons of increasing size are shown below. Suppose the dot pattern continues so that each successive hexagon contains one more band of dots. How many dots are in the next hexagon?



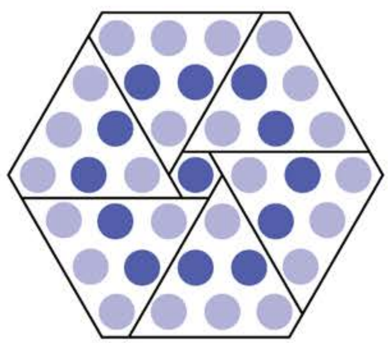
(A) 35 **(B)** 37 **(C)** 39 **(D)** 43 **(E)** 49

Answer (B): The fourth hexagon has 3 bands of dots surrounding a center dot, as shown below. The innermost band has $6 \cdot 1 = 6$ dots, the middle band has $6 \cdot 2 = 12$ dots, and the outermost band has $6 \cdot 3 = 18$ dots. Therefore the total number of dots in the fourth hexagon is $1 + 6 + 12 + 18 = 37$.



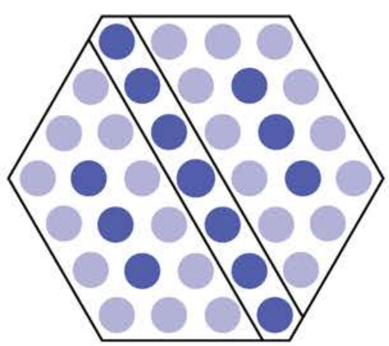
OR

Excluding the center dot, each hexagon can be subdivided into six congruent triangular regions, as shown in the figure below. In the fourth hexagon, each triangular region contains $1 + 2 + 3 = 6$ dots, so the total number of dots in the fourth hexagon is $1 + 6 \cdot 6 = 37$.



OR

In each hexagon, the dots that lie on a diagonal form a line of symmetry, as shown in the figure below. The fourth hexagon has 7 dots on a diagonal and $6 + 5 + 4 = 15$ dots on either side, for a total of $7 + 2 \cdot 15 = 37$ dots.



Note: The number of dots in each hexagon is called a *hex number*. Generalizing from the first and second solutions, the nth hex number can be calculated as

$$1 + 6 \cdot (1 + 2 + 3 + \cdots + (n-1)) = 1 + 6 \cdot \frac{n(n-1)}{2} = 1 + 3n(n-1).$$

Generalizing from the third solution, the nth hex number equals

$$(2n-1) + 2 \cdot ((2n-2) + (2n-3) + \cdots + n) = 1 + 3n(n-1).$$

It can be shown that the sum of the first n hex numbers is n^3. For example, the sum of the first 2 hex numbers is $1 + 7 = 8 = 2^3$, the sum of the first 3 hex numbers is $1 + 7 + 19 = 27 = 3^3$, and the sum of the first 4 hex numbers is $1 + 7 + 19 + 37 = 64 = 4^3$.

5. Three fourths of a pitcher is filled with pineapple juice. The pitcher is emptied by pouring an equal amount of juice into each of 5 cups. What percent of the total capacity of the pitcher did each cup receive?

(A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 25

Answer (C): Each cup received one fifth of three fourths of the pitcher. This corresponds to a percentage of

$$\frac{1}{5} \cdot \frac{3}{4} = \frac{3}{20} = \frac{15}{100} = 15\%.$$

OR

The juice filled up 75% of the pitcher, so each of the 5 cups received

$$\frac{1}{5} \cdot 75\% = 15\%.$$

6. Aaron, Darren, Karen, Maren, and Sharon rode on a small train that has five cars that seat one person each. Maren sat in the last car. Aaron sat directly behind Sharon. Darren sat in one of the cars in front of Aaron. At least one person sat between Karen and Darren. Who sat in the middle car?

(A) Aaron **(B)** Darren **(C)** Karen **(D)** Maren **(E)** Sharon

Answer (A): It is given that Maren sat in the last car, Aaron sat directly behind Sharon, and Darren sat in one of the cars in front of Aaron. Based on these three conditions, the possible seating arrangements are:

D S A _ M

_ D S A M

D _ S A M

The last condition states that Karen and Darren did not sit in adjacent cars, which eliminates the second and third possibilities. That leaves the first arrangement as the only one that satisfies all of the conditions, with Karen sitting in the fourth car and Aaron in the middle car.

7. How many integers between 2020 and 2400 have four distinct digits arranged in increasing order? (For example, 2357 is one such integer.)

   (A) 9 **(B)** 10 **(C)** 15 **(D)** 21 **(E)** 28

   Answer (C): The digits are in increasing order so the first digit must be 2, the second digit must be 3, and the third digit must be 4, 5, 6, 7, or 8. The possible integers are listed below.

Leading Digits	Units Digit
234_	5, 6, 7, 8, or 9
235_	6, 7, 8, or 9
236_	7, 8, or 9
237_	8 or 9
238_	9

   The total is $5+4+3+2+1=15$ integers.

   OR

   The first digit must be 2, the second digit must be 3, and the remaining two digits must be chosen from among the six digits 4, 5, 6, 7, 8, and 9. There are ${}_6C_2 = \binom{6}{2} = \frac{6!}{4!2!} = \frac{6\cdot 5}{2} = 15$ pairs of those digits and only one possible order for each pair, so there are 15 such integers.

8. Ricardo has 2020 coins, some of which are pennies (1-cent coins) and the rest of which are nickels (5-cent coins). He has at least one penny and at least one nickel. What is the difference in cents between the greatest possible and least possible amounts of money that Ricardo can have?

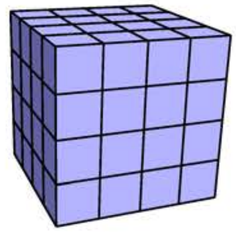
   (A) 8062 **(B)** 8068 **(C)** 8072 **(D)** 8076 **(E)** 8082

   Answer (C): To obtain the greatest possible amount of money, use as many nickels as possible: 2019 nickels and 1 penny. This gives $2019\cdot 5 + 1\cdot 1 = 10{,}096$ cents. Similarly, to obtain the least possible amount of money, use as many pennies as possible: 1 nickel and 2019 pennies. This gives $1\cdot 5 + 2019\cdot 1 = 2024$ cents. Therefore the difference between the greatest possible and least possible amounts of money is $10{,}096 - 2024 = 8072$ cents.

   OR

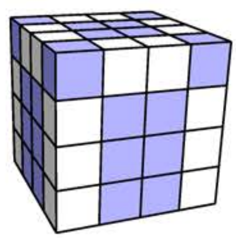
   One penny and one nickel will be used in either case so they can be removed, leaving 2018 coins. The greatest possible sum will result from having 2018 nickels, and the least possible sum will result from having 2018 pennies. The difference in amounts is $2018\cdot 5 - 2018\cdot 1 = 2018\cdot 4 = 8072$ cents.

9. Akash's birthday cake is in the form of a $4 \times 4 \times 4$ inch cube. The cake has icing on the top and the four side faces, and no icing on the bottom. Suppose the cake is cut into 64 smaller cubes, each measuring $1 \times 1 \times 1$ inch, as shown below. How many of the small pieces will have icing on exactly two sides?



(A) 12 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 24

Answer (D):



The cuts divide the cake into four $4 \times 4 \times 1$ inch horizontal layers. In the top layer there are 8 pieces having exactly two sides with icing, namely the non-corner edge pieces that are part of both the top face and a side face. In each of the next 3 layers, the 4 corner pieces are the only ones having two sides with icing. This gives a total of $8 + 3 \cdot 4 = 20$ small pieces with icing on exactly two sides.

OR

First consider the 8 edges of the cake that are not along the bottom face. Each of those edges has 2 middle pieces with icing on exactly two sides. In addition to these pieces, the 4 corner pieces of the bottommost layer also have two sides with icing. This gives a total of $8 \cdot 2 + 4 = 20$ small pieces with icing on exactly two sides.

10. Zara has a collection of 4 marbles: an Aggie, a Bumblebee, a Steelie, and a Tiger. She wants to display them in a row on a shelf, but does not want to put the Steelie and the Tiger next to one another. In how many ways can she do this?

(A) 6 **(B)** 8 **(C)** 12 **(D)** 18 **(E)** 24

Answer (C): There are 6 ways Zara can place the Steelie and the Tiger so that they are not next to one another. Those arrangements are shown below.

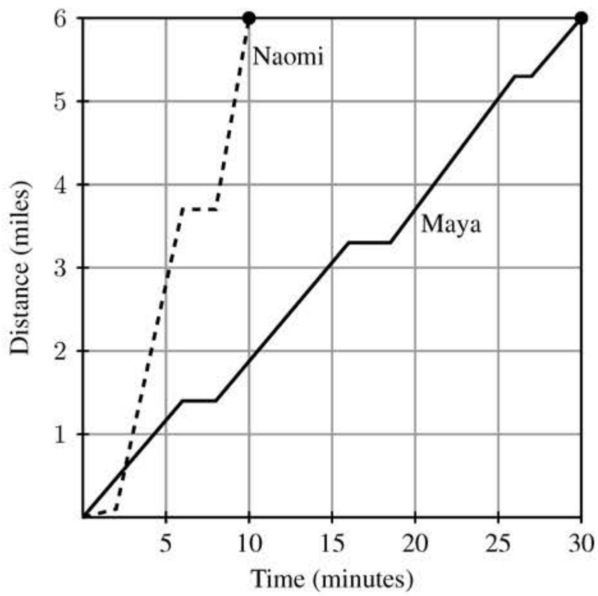
S _ T _	T _ S _
S _ _ T	T _ _ S
_ S _ T	_ T _ S

For each arrangement, there are 2 ways to place the remaining marbles: either the Aggie appears to the left of the Bumblebee or it appears to the right. This makes $6 \cdot 2 = 12$ arrangements with the Steelie and the Tiger not next to one another.

OR

The number of arrangements can be determined by using complementary counting. That is, begin with the total number of possible arrangements, which is $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, then subtract the number of ways to place the Steelie and the Tiger next to each other. If the Steelie and the Tiger are considered as a single object, then there are 3 objects to arrange and $3! = 3 \cdot 2 \cdot 1 = 6$ ways to do so. There are 2 ways to arrange the Steelie and the Tiger, so there are $6 \cdot 2 = 12$ arrangements with the Steelie and the Tiger adjacent to each other. Thus there are $24 - 12 = 12$ arrangements with the Steelie and the Tiger not next to one another.

11. After school, Maya and Naomi headed to the beach, 6 miles away. Maya decided to bike while Naomi took a bus. The graph below shows their journeys, indicating the time and distance traveled. What was the difference, in miles per hour, between Naomi's and Maya's average speeds?



(A) 6 **(B)** 12 **(C)** 18 **(D)** 20 **(E)** 24

Answer (E): Average speed equals the total distance traveled divided by the elapsed time. Maya covered the distance of 6 miles in 30 minutes, which is $\frac{1}{2}$ of an hour. Her average speed was $6 \div \frac{1}{2} = 12$ miles per hour. Naomi covered the same distance in 10 minutes, which is $\frac{1}{6}$ of an hour. Her average speed was $6 \div \frac{1}{6} = 36$ miles per hour. The difference between their average speeds was $36 - 12 = 24$ miles per hour.

12. For positive integer n, the factorial notation $n!$ represents the product of the integers from n to 1. (For example, $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$.) What value of N satisfies the following equation?

$$5! \cdot 9! = 12 \cdot N!$$

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Answer (A): The given equation implies that

$$N! = \frac{5! \cdot 9!}{12} = \frac{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (9!)}{12} = 10 \cdot 9! = 10!,$$

so $N = 10$.

13. Jamal has a drawer containing 6 green socks, 18 purple socks, and 12 orange socks. After adding more purple socks, Jamal noticed that there is now a 60% chance that a sock randomly selected from the drawer is purple. How many purple socks did Jamal add?

(A) 6 **(B)** 9 **(C)** 12 **(D)** 18 **(E)** 24

Answer (B): A 60% chance of choosing a purple sock implies a 40% chance of choosing a green or orange sock. It follows that the $6 + 12 = 18$ non-purple socks represent 40% of all the socks. This means that 9 socks represent 20% of the total, and 27 socks represent 60% of the total. Jamal therefore added $27 - 18 = 9$ purple socks to the drawer.

OR

Let x equal the number of additional purple socks, increasing the number of purple socks to $18 + x$ and the total number of socks to $36 + x$. Because the proportion of purple socks must equal 60%, the following equation must be satisfied:

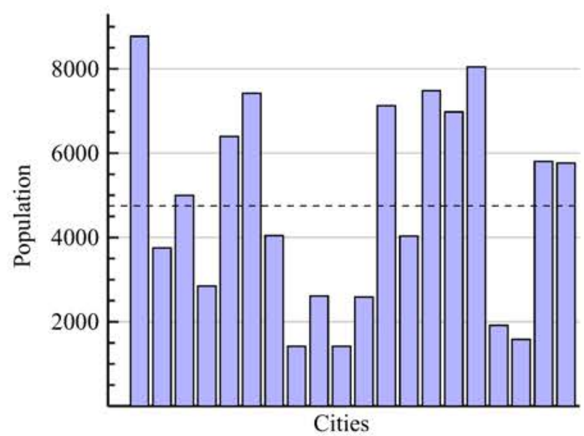
$$\frac{18 + x}{36 + x} = \frac{60}{100} = \frac{3}{5}.$$

Solving for x yields

$$\begin{aligned} 5(18 + x) &= 3(36 + x) \\ 90 + 5x &= 108 + 3x \\ 2x &= 18 \\ x &= 9. \end{aligned}$$

Jamal added 9 purple socks to the drawer.

14. There are 20 cities in the County of Newton. Their populations are shown in the bar chart below. The average population of all the cities is indicated by the horizontal dashed line. Which of the following is closest to the total population of all 20 cities?



(A) 65,000 **(B)** 75,000 **(C)** 85,000 **(D)** 95,000 **(E)** 105,000

Answer (D): The total population of all the cities equals the number of cities multiplied by the average city population. There are 20 cities, and the dashed line shows that the average population is between 4500 and 5000. It follows that the total population of the 20 cities is between $20 \cdot 4500 = 90{,}000$ and $20 \cdot 5000 = 100{,}000$. Only choice **(D)** 95,000 is within 5000 of the actual total population.

15. Suppose 15% of x equals 20% of y. What percentage of x is y ?

(A) 5 **(B)** 35 **(C)** 75 **(D)** $133\frac{1}{3}$ **(E)** 300

Answer (C): First note that $15\% = 0.15$, and $20\% = 0.20$. It is given that 15% of x equals 20% of y, so

$$0.15x = 0.20y.$$

Solving for y yields

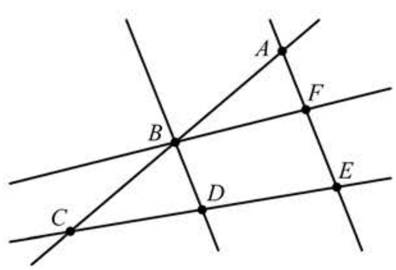
$$y = \frac{0.15}{0.20}x = \frac{15}{20}x = \frac{75}{100}x.$$

Therefore y is 75% of x.

OR

Let $x = 100$. Then 15% of x is 15. This equals 20% of y, which is the same as $\frac{1}{5}y$. Solving the equation $15 = \frac{1}{5}y$ yields $y = 5 \cdot 15 = 75$. Therefore y is 75% of x.

16. Each of the points A, B, C, D, E, and F in the figure below represents a different digit from 1 to 6. Each of the five lines shown passes through some of these points. The digits along each line are added to produce five sums, one for each line. The total of the five sums is 47. What is the digit represented by B ?



(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Answer (E): The total of the five sums is

$$\begin{aligned}(A + B + C) + (A + F + E) + (B + D) + (B + F) + (C + D + E)\\ = 2A + 3B + 2C + 2D + 2E + 2F\\ = 2(A + B + C + D + E + F) + B,\end{aligned}$$

which equals 47. The expression $A + B + C + D + E + F$ equals 21, the sum of the digits from 1 to 6, so $2 \cdot 21 + B = 47$. Therefore $B = 47 - 42 = 5$.

OR

Note that every point lies on exactly two lines except for B, which lies on three lines. Because the sum of the digits from 1 to 6 is 21, this gives a total sum of $2 \cdot 21 + B = 47$. Therefore $B = 47 - 42 = 5$.

17. How many factors of 2020 have more than 3 factors? (As an example, 12 has 6 factors, namely $1, 2, 3, 4, 6$, and 12.)

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Answer (B): The number 2020 has 12 factors.

$$\begin{aligned} 2020 &= 1 \cdot 2020 \\ &= 2 \cdot 1010 \\ &= 4 \cdot 505 \\ &= 5 \cdot 404 \\ &= 10 \cdot 202 \\ &= 20 \cdot 101 \end{aligned}$$

These factors may be classified as follows:

- The number 1 has exactly 1 factor.
- The numbers $2, 5$, and 101 are primes, so each has exactly 2 factors.
- The number 4 has exactly 3 factors, namely $1, 2$, and 4.
- The remaining 7 numbers, 10, 20, 202, 404, 505, 1010, and 2020, each have more than 3 factors.

Thus 7 of the factors of 2020 have more than 3 factors.

OR

In order to determine the number of factors of 2020, note that the prime factorization of 2020 is

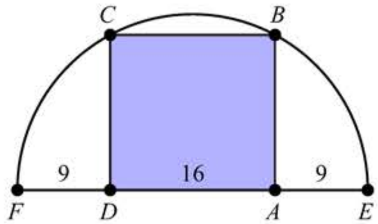
$$2020 = 2^2 \cdot 5^1 \cdot 101^1.$$

This means that each factor of 2020 has the form $2^a \cdot 5^b \cdot 101^c$, where $a = 0, 1,$ or 2; $b = 0$ or 1; and $c = 0$ or 1. Thus the number of factors of 2020 is $3 \cdot 2 \cdot 2 = 12$.

By similar reasoning, any positive integer with at least 2 distinct prime factors, say p and q, has at least 4 factors, namely 1, p, q, and pq.

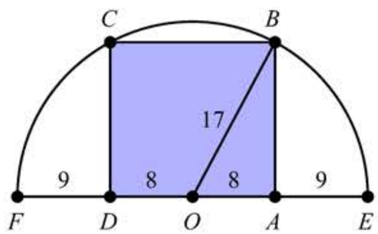
The factors of 2020 that have no more than 1 distinct prime factor are 1, 2, 4, 5, and 101. Each of these has at most 3 factors. The remaining 7 factors, 10, 20, 202, 404, 505, 1010, and 2020, each have at least 2 distinct prime factors. Therefore 7 of the factors of 2020 have more than 3 factors.

18. Rectangle $ABCD$ is inscribed in a semicircle with diameter $\overline{FE}$, as shown in the figure. Let $DA = 16$, and let $FD = AE = 9$. What is the area of $ABCD$?



(A) 240 **(B)** 248 **(C)** 256 **(D)** 264 **(E)** 272

Answer (A):



Let O be the center of the circle. Then $DO = OA = 8$ and radius $OE = OA + AE = 8 + 9 = 17$. Draw radius $\overline{OB}$ which also has length 17. Then $\triangle OAB$ is a right triangle with hypotenuse 17 and base leg 8. By the Pythagorean Theorem $AB = \sqrt{17^2 - 8^2} = \sqrt{289 - 64} = \sqrt{225} = 15$. The area of rectangle $ABCD$ is $DA \cdot AB = 16 \cdot 15 = 240$.

19. A number is called *flippy* if its digits alternate between two distinct digits. For example, 2020 and 37373 are flippy, but 3883 and 123123 are not. How many five-digit flippy numbers are divisible by 15 ?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 8

Answer (B): A five-digit flippy number looks like $\underline{A}\,\underline{B}\,\underline{A}\,\underline{B}\,\underline{A}$ for some distinct digits A and B with $A \neq 0$ (otherwise it would not be a five-digit number). In order for this number to be divisible by 15, it must be divisible by 5 and 3.

Any number divisible by 5 will have a units digit of 0 or 5. Because $A \neq 0$, A must equal 5. This means the flippy number must have the form $\underline{5}\,\underline{B}\,\underline{5}\,\underline{B}\,\underline{5}$.

For a number to be divisible by 3, the sum of its digits must be divisible by 3. The sum of the digits of $\underline{5}\,\underline{B}\,\underline{5}\,\underline{B}\,\underline{5}$ is $15 + 2B$. This means $2B$ must be divisible by 3, so B is divisible by 3. There are 4 possible digits for B: 0, 3, 6, or 9. Therefore there are 4 five-digit flippy numbers divisible by 15: 50505, 53535, 56565, and 59595.

20. A scientist walking through a forest recorded as integers the heights of 5 trees standing in a row. She observed that each tree was either twice as tall or half as tall as the one to its right. Unfortunately some of her data was lost when rain fell on her notebook. Her notes are shown below, with blanks

indicating the missing numbers. Based on her observations, the scientist was able to reconstruct the lost data. What was the average height of the trees, in meters?

Tree 1	___ meters
Tree 2	11 meters
Tree 3	___ meters
Tree 4	___ meters
Tree 5	___ meters
Average height	___.2 meters

(A) 22.2 **(B)** 24.2 **(C)** 33.2 **(D)** 35.2 **(E)** 37.2

Answer (B): The notebook shows that the second tree is 11 meters tall. The first and third trees must be twice as tall at 22 meters, otherwise if they were half as tall, their heights would not be integers. This means the first 3 trees have a total height of $22 + 11 + 22 = 55$ meters. Because the third tree is 22 meters tall, the only possibilities for the heights of the last 2 trees are

- 11 and 22 meters, which result in an average of $\frac{55+11+22}{5} = 17.6$ meters,
- 44 and 22 meters, which result in an average of $\frac{55+44+22}{5} = 24.2$ meters, or
- 44 and 88 meters, which result in an average of $\frac{55+44+88}{5} = 37.4$ meters.

The average ends in ".2" so the second option must be the correct one. Thus it must be that the heights of the trees are $22, 11, 22, 44,$ and 22 meters, and the average height of all the trees is 24.2 meters.

OR

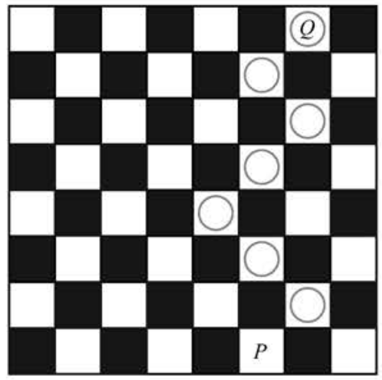
The average tree height equals the sum of the tree heights divided by 5. Because the average ends in ".2" which equals $\frac{2}{10} = \frac{1}{5}$, the sum of the tree heights must be one more than a multiple of 5.

As shown above, the heights of the first 3 trees are 22, 11, and 22 meters, producing a total height of 55 meters, which is evenly divisible by 5. The sum of the heights of the last 2 trees must be one more than a multiple of 5. Because the third tree is 22 meters tall, the only possibilities for the heights of the last 2 trees are

- 11 and 22 meters, which sum to 33 meters,
- 44 and 22 meters, which sum to 66 meters, or
- 44 and 88 meters, which sum to 132 meters.

Only the second option is one more than a multiple of 5. Thus it must be that the heights of the trees are $22, 11, 22, 44,$ and 22 meters, and the average height is $\frac{55+44+22}{5} = 24.2$ meters.

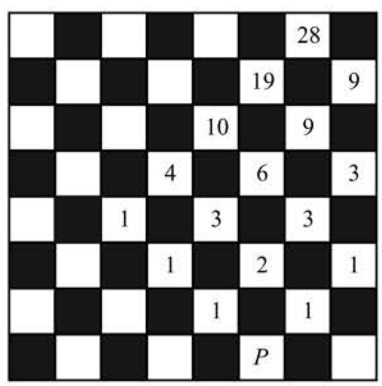
21. A game board consists of 64 squares that alternate in color between black and white. The figure below shows square P in the bottom row and square Q in the top row. A marker is placed at P. A *step* consists of moving the marker onto one of the adjoining white squares in the row above. How many 7-step paths are there from P to Q ? (The figure shows a sample path.)



(A) 28 **(B)** 30 **(C)** 32 **(D)** 33 **(E)** 35

Answer (A):

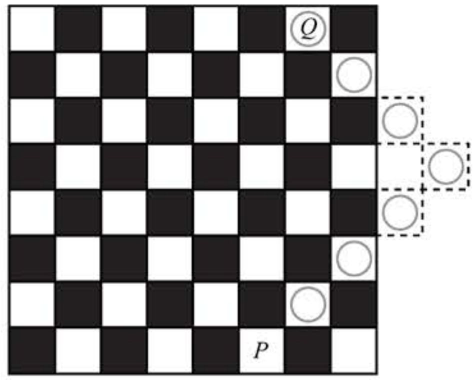
Beginning with the row above P and filling in one row at a time, the number of paths that lead from P to each square can be counted, as shown in the figure below. (Squares that cannot be reached or do not lead to Q in 7 moves are left blank.) Note that the number of paths leading to a square equals the sum of the counts in the adjoining squares below it. (For example, to reach the square with 10 paths, it is necessary to pass through one of the adjoining squares below; they have 4 and 6 paths leading to them, respectively.) Applying this counting method to each row, moving up from the bottom, results in a total of 28 seven-step paths from P to Q.



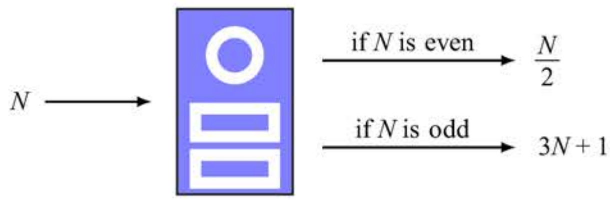
OR

Each step moves the marker either one column to the left (L) or one column to the right (R). Because Q is one column to the right of P, each path must consist of 4 moves to the right and 3 moves to the left. Ignoring the possibility that a path may extend past the right edge of the board, the number of paths from P to Q is equivalent to the number of distinguishable permutations of $RRRRLLL$, which equals the number of combinations of 7 objects taken 4 at a time: ${}_7C_4 = \binom{7}{4} = \frac{7!}{4!3!} = 35$.

This total, however, includes paths that move the marker off the right edge of the board. One such path is shown below. A path leaves the board when the number of R moves exceeds the number of L moves by 3. There are 7 paths that extend beyond the board: $\underline{LRRRR}LL$, $\underline{RLRRR}LL$, $\underline{RRLRR}LL$, and the 4 paths that begin with $\underline{RRR}$. Removing these 7 paths leaves $35 - 7 = 28$ seven-step paths from P to Q.



22. When a positive integer N is fed into a machine, the output is a number calculated according to the rule shown below.



For example, starting with an input of $N = 7$, the machine will output $3 \cdot 7 + 1 = 22$. Then if the output is repeatedly inserted into the machine five more times, the final output is 26.

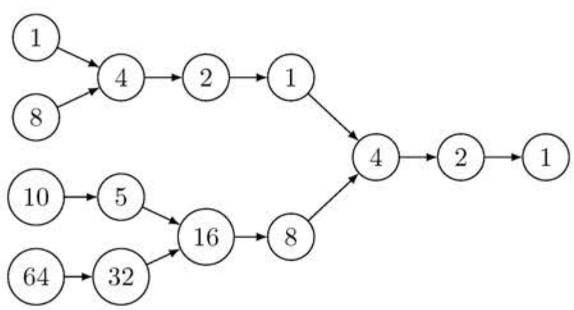
$$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26$$

When the same 6-step process is applied to a different starting value of N, the final output is 1. What is the sum of all such integers N?

$$N \to \_\_ \to \_\_ \to \_\_ \to \_\_ \to \_\_ \to 1$$

(A) 73 **(B)** 74 **(C)** 75 **(D)** 82 **(E)** 83

Answer (E): Working backwards from the number 1, as shown below, there are 4 possible starting values of N: $1, 8, 10$, and 64. The sum of these values is $1 + 8 + 10 + 64 = 83$.



Note: The Collatz Conjecture states that the number 1 will appear eventually no matter which positive integer N is chosen as the starting value. For example if the $N = 7$ sequence is extended past 26, the outputs will be

$$26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1.$$

It is not known whether this conjecture is true.

23. Five different awards are to be given to three students. Each student will receive at least one award. In how many different ways can the awards be distributed?

(A) 120 **(B)** 150 **(C)** 180 **(D)** 210 **(E)** 240

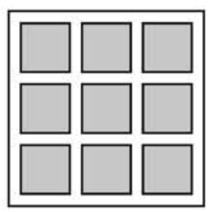
Answer (B): There are two cases to consider.

Case 1: One student receives 1 award and the other two students receive 2 awards each. There are 3 ways to select the student who will receive 1 award and 5 choices for that student's award. That leaves 4 awards to distribute evenly. For the next student there are ${}_4C_2 = \binom{4}{2} = \frac{4\cdot 3}{2} = 6$ ways to choose that student's 2 awards, leaving the remaining 2 awards for the third student. Therefore the number of ways to give 1 award to one student and 2 awards to each of the other two students is $3 \cdot 5 \cdot 6 = 90$.

Case 2: One student receives 3 awards and the other two students receive 1 award each. There are 3 ways to select the student who receives 3 awards and ${}_5C_3 = \binom{5}{3} = \frac{5\cdot 4\cdot 3}{3\cdot 2\cdot 1} = 10$ ways to choose that student's awards. The remaining 2 awards can be distributed to the other two students in 2 ways. Therefore the number of ways to give 3 awards to one student and 1 award to each of the other two students is $3 \cdot 10 \cdot 2 = 60$.

In total there are $90 + 60 = 150$ ways to distribute the 5 awards.

24. A large square region is paved with n^2 gray square tiles, each measuring s inches on a side. A border d inches wide surrounds each tile. The figure below shows the case for $n = 3$. When $n = 24$, the 576 gray tiles cover 64% of the area of the large square region. What is the ratio $\frac{d}{s}$ for this larger value of n?



(A) $\frac{6}{25}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{9}{25}$ **(D)** $\frac{7}{16}$ **(E)** $\frac{9}{16}$

Answer (A): When $n = 24$, there are $24^2 = 576$ gray tiles, each with an area of s^2 square inches. Together the tiles cover an area of 24^2s^2 square inches. Each side of the large square region spans 24 tiles plus 25 borders for a side length of $24s + 25d$ inches, giving a total area of $(24s + 25d)^2$ square inches. It is given that the gray tiles cover 64% of the total area. It follows that

$$\frac{\text{area of gray tiles}}{\text{area of large square}} = \frac{24^2s^2}{(24s+25d)^2} = \frac{64}{100} = \frac{16}{25},$$

which simplifies to

$$\frac{24s}{24s+25d} = \frac{4}{5}.$$

Solving the equation for $\frac{d}{s}$ produces the following result:

$$\frac{24s+25d}{24s} = \frac{5}{4}$$

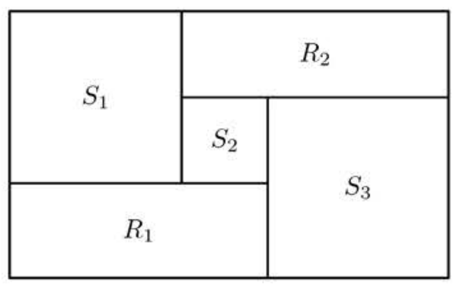
$$\begin{aligned} 1 + \frac{25d}{24s} &= 1 + \frac{1}{4} \\ \frac{d}{s} &= \frac{24}{25} \cdot \frac{1}{4} \\ &= \frac{6}{25}. \end{aligned}$$

Therefore, when $n = 24$, the ratio $\frac{d}{s}$ is $\frac{6}{25}$.

OR

Let the side length of each gray tile be $s = 1$ inch. Then the tiles have a total area of 576 square inches. Because 64% of the large square region is covered by gray tiles, the large square region has an area of $\frac{576}{0.64} = 900$ square inches, and hence its side length is 30 inches. Each side of the large square region spans 24 tiles plus 25 borders, so $24 + 25d = 30$ and $d = \frac{30-24}{25} = \frac{6}{25}$. The desired ratio is $\frac{d}{s} = d = \frac{6}{25}$.

25. Rectangles R_1 and R_2, and squares S_1, S_2, and S_3, shown below, combine to form a rectangle that is 3322 units wide and 2020 units high. What is the side length of S_2 in units?



(A) 651 **(B)** 655 **(C)** 656 **(D)** 662 **(E)** 666

Answer (A): Let x_1, x_2, and x_3 represent the side lengths of squares S_1, S_2, and S_3, respectively. Then the width of the large rectangle equals

$$x_1 + x_2 + x_3 = 3322$$

and the height of the large rectangle equals

$$x_1 - x_2 + x_3 = 2020.$$

Subtracting the two equations yields $2x_2 = 1302$, so $x_2 = 651$. The side length of S_2 is therefore 651 units.

Note: Although the size of square S_2 is fixed, the sizes of S_1 and S_3 can vary as long as the sum of their side lengths is $x_1 + x_3 = 3322 - x_2 = 2671$.

Problems and Solutions contributed by Hannah Alpert, Bela Bajnok, Helen Beylkin, Silva Chang, Barbara Currier, Dotty Dady, Tivadar Diveki, Steven Dunbar, Marta Eso, Chris Jeuell, Steven Klee, Rich Morrow, Bryan Nevarez, Daheng Shen, Jeganathan Sriskandarajah, Zsuzsanna Szaniszlo, David Wells, Carl Yerger, and Jesse Zhang.

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