MAA AMC

American Mathematics Competitions

Official Solutions

MAA American Mathematics Competitions

35th Annual

AMC 8

Tuesday, November 12, 2019 through Monday, November 18, 2019

These official solutions give at least one solution for each problem on this year's competition and show that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

Teachers are encouraged to share these solutions with their students. Copies of the problem booklet and official solutions may be shared with your students for educational purposes. However, the publication, reproduction, or communication of the problems or solutions for this competition with anyone outside the classroom is a violation of the competition rules. This includes dissemination via copier, telephone, email, Internet, or media of any type.

Questions and comments about this competition should be sent to:

amcinfo@maa.org

or

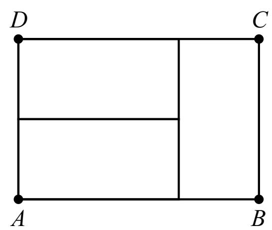
MAA American Mathematics Competitions
PO Box 471
Annapolis Junction, MD 20701

1. Ike and Mike go into a sandwich shop with a total of \$30.00 to spend. Sandwiches cost \$4.50 each and soft drinks cost \$1.00 each. Ike and Mike plan to buy as many sandwiches as they can and use any remaining money to buy soft drinks. Counting both soft drinks and sandwiches, how many items will they buy?

   (A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

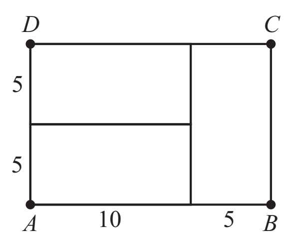
   Answer (D): Ike and Mike can buy 2 sandwiches for \$9.00, so they can buy 6 sandwiches for \$27.00. This leaves them with \$3.00, which is not enough to buy another sandwich, but it is enough to buy 3 soft drinks. The total number of items they will buy is $6 + 3 = 9$.

2. Three identical rectangles are put together to form rectangle $ABCD$, as shown in the figure below. Given that the length of the shorter side of each of the smaller rectangles is 5 feet, what is the area in square feet of rectangle $ABCD$?



   (A) 45 **(B)** 75 **(C)** 100 **(D)** 125 **(E)** 150

   Answer (E): The length of side $\overline{AD}$ is 10 feet because it is the sum of the lengths of the shorter sides of two of the small rectangles. It follows that the small rectangles are 5 feet by 10 feet. Therefore rectangle $ABCD$ is 10 feet by 15 feet with an area of 150 square feet.



3. Which of the following is the correct order of the fractions $\frac{15}{11}$, $\frac{19}{15}$, and $\frac{17}{13}$, from least to greatest?

   (A) $\frac{15}{11} < \frac{17}{13} < \frac{19}{15}$ **(B)** $\frac{15}{11} < \frac{19}{15} < \frac{17}{13}$ **(C)** $\frac{17}{13} < \frac{19}{15} < \frac{15}{11}$

   (D) $\frac{19}{15} < \frac{15}{11} < \frac{17}{13}$ **(E)** $\frac{19}{15} < \frac{17}{13} < \frac{15}{11}$

   Answer (E): To determine the order of $\frac{19}{15}$ and $\frac{17}{13}$, rewrite the fractions using a common denominator: $\frac{19 \cdot 13}{15 \cdot 13}$ and $\frac{17 \cdot 15}{13 \cdot 15}$. Because

$$19 \cdot 13 = (16 + 3)(16 - 3) = 16^2 - 3^2,$$

$$17 \cdot 15 = (16+1)(16-1) = 16^2 - 1^2,$$

and $16^2 - 3^2 < 16^2 - 1^2$, it follows that $\frac{19}{15} < \frac{17}{13}$.

Similarly, to determine the order of $\frac{17}{13}$ and $\frac{15}{11}$, rewrite the fractions using a common denominator: $\frac{17 \cdot 11}{13 \cdot 11}$ and $\frac{15 \cdot 13}{11 \cdot 13}$. Because

$$17 \cdot 11 = (14+3)(14-3) = 14^2 - 3^2,$$
$$15 \cdot 13 = (14+1)(14-1) = 14^2 - 1^2,$$

and $14^2 - 3^2 < 14^2 - 1^2$, it follows that $\frac{17}{13} < \frac{15}{11}$.

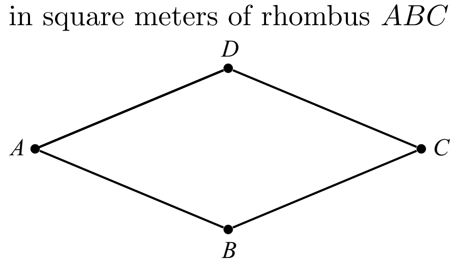
OR

Subtracting 1 from each fraction results in the fractions $\frac{4}{11}$, $\frac{4}{15}$, and $\frac{4}{13}$. Because $\frac{4}{15} < \frac{4}{13} < \frac{4}{11}$, it follows that $\frac{19}{15} < \frac{17}{13} < \frac{15}{11}$.

OR

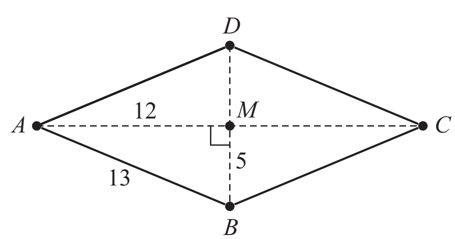
Given a fraction $\frac{a}{b}$ where $0 < b < a$, if n is a positive integer, then $bn < an$ and so $b(a+n) = ab + bn < ab + an = a(b+n)$. Thus $\frac{a+n}{b+n} < \frac{a}{b}$. Therefore $\frac{19}{15} < \frac{17}{13} < \frac{15}{11}$.

4. Quadrilateral $ABCD$ is a rhombus with perimeter 52 meters. The length of diagonal $\overline{AC}$ is 24 meters. What is the area in square meters of rhombus $ABCD$?



(A) 60 **(B)** 90 **(C)** 105 **(D)** 120 **(E)** 144

Answer (D): Let M be the midpoint of $\overline{AC}$. Then $AM = 12$. Because the diagonals of a rhombus are perpendicular and bisect each other, $\overline{BM}$ is perpendicular to $\overline{AC}$. Because all four sides of a rhombus have the same length, $AB = \frac{52}{4} = 13$. By the Pythagorean Theorem, $BM = \sqrt{13^2 - 12^2} = 5$. The area of $\triangle ABM$ is $\frac{1}{2} \cdot 12 \cdot 5 = 30$ square meters. Thus the area of rhombus $ABCD$ is $4 \cdot 30 = 120$ square meters.

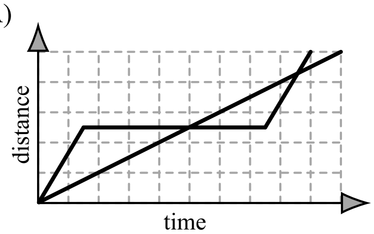


OR

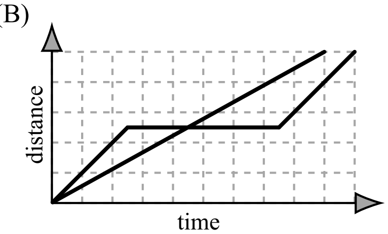
As above, $BM = 5$, and so $BD = 10$. The area of a rhombus is one-half the product of the lengths of the diagonals. The area of $ABCD$ is therefore $\frac{1}{2} \cdot 24 \cdot 10 = 120$ square meters.

5. A tortoise challenges a hare to a race. The hare eagerly agrees and quickly runs ahead, leaving the slow-moving tortoise behind. Confident that he will win, the hare stops to take a nap. Meanwhile, the tortoise walks at a slow steady pace for the entire race. The hare awakes and runs to the finish line, only to find the tortoise already there. Which of the following graphs matches the description of the race, showing the distance d traveled by the two animals over time t from start to finish?

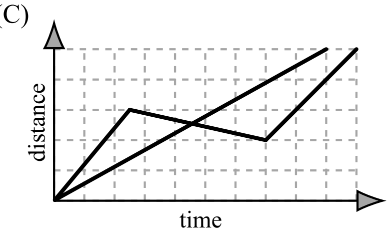
(A)



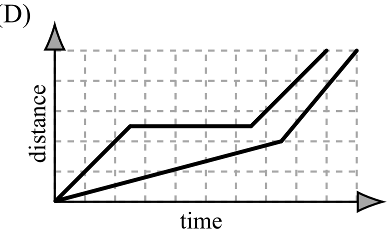
(B)



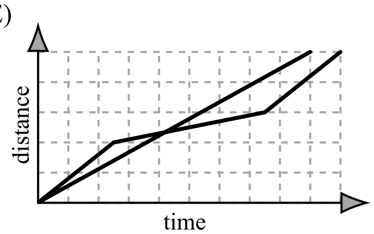
(C)



(D)

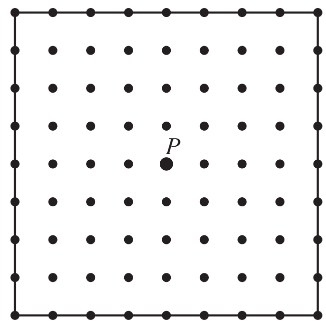


(E)



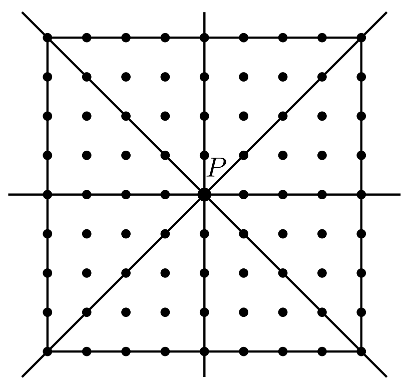
Answer (B): The tortoise moves at a slow and steady speed. The graph of the tortoise's journey corresponds to a straight line. Graph (D) is eliminated because there is no straight line. The hare's journey consists of running, then napping, then running again. The graph of the hare's journey corresponds to a line with steeper slope than the tortoise's, a horizontal line during his nap when his distance does not change, and another steep line. Graphs (C) and (E) are eliminated because there is no horizontal line corresponding to the hare's nap. The tortoise reaches the finish line before the hare. Graph (A) is eliminated because it shows the hare catching up to the tortoise and finishing first. Only graph (B) matches the description of the problem.

6. There are 81 grid points (uniformly spaced) in the square shown in the diagram below, including the points on the edges. Point P is the center of the square. Given that point Q is randomly chosen from among the other 80 points, what is the probability that the line PQ is a line of symmetry for the square?



(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{9}{20}$ **(E)** $\frac{1}{2}$

Answer (C): The square has 4 lines of symmetry, each containing P and 8 more grid points. The probability that Q is chosen so that line PQ is a line of symmetry is therefore $\frac{4 \cdot 8}{80} = \frac{32}{80} = \frac{2}{5}$.



7. Shauna takes five tests, each worth a maximum of 100 points. Her scores on the first three tests are 76, 94, and 87. In order to average 81 for all five tests, what is the lowest score she could earn on one of the other two tests?

(A) 48 **(B)** 52 **(C)** 66 **(D)** 70 **(E)** 74

Answer (A): To average 81 for all five tests, Shauna would have to earn a total of $5 \cdot 81 = 405$ points. So far, she has earned $76 + 94 + 87 = 257$ points, so she would need to earn $405 - 257 = 148$ points on the remaining two tests. The most she could score on one test is 100 points, so the lowest score she could earn on the other test is $148 - 100 = 48$ points.

8. Gilda has a bag of marbles. She gives 20% of them to her friend Pedro. Then Gilda gives 10% of what is left to another friend, Ebony. Finally, Gilda gives 25% of what is now left in the bag to her brother Jimmy. What percentage of her original bag of marbles does Gilda have left for herself?

   (A) 20 **(B)** $33\frac{1}{3}$ **(C)** 38 **(D)** 45 **(E)** 54

   Answer (E): Imagine that Gilda starts with 100 marbles. She first gives 20% of the 100 marbles to Pedro, leaving her with 80. She then gives 10% of the 80 marbles to Ebony, leaving her with $80 - 8 = 72$. Finally Gilda gives 25% of the 72 marbles to Jimmy, leaving her with $\frac{3}{4} \cdot 72 = 54$. Thus Gilda ends up with $\frac{54}{100}$, which is 54% of the marbles.

   OR

   After the three gifts, Gilda is left with three-fourths of nine-tenths of four-fifths of the marbles she started with, and $\frac{3}{4} \cdot \frac{9}{10} \cdot \frac{4}{5} = \frac{27}{50}$, which is 54%.

9. Alex and Felicia each have cats as pets. Alex buys cat food in cylindrical cans that are 6 cm in diameter and 12 cm high. Felicia buys cat food in cylindrical cans that are 12 cm in diameter and 6 cm high. What is the ratio of the volume of one of Alex's cans to the volume of one of Felicia's cans?

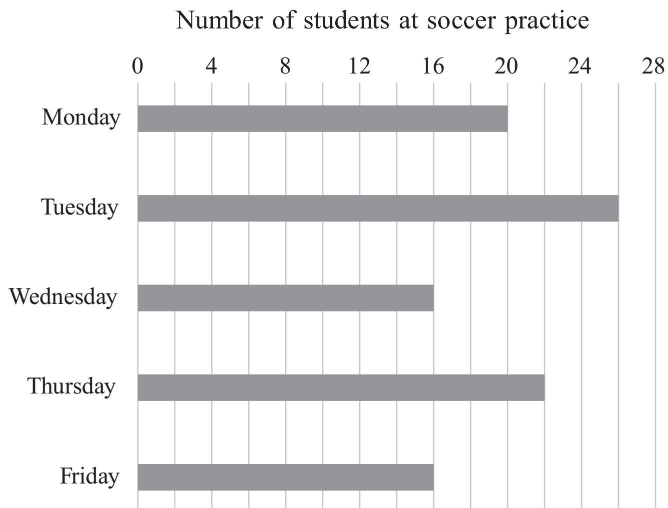
   (A) $1:4$ **(B)** $1:2$ **(C)** $1:1$ **(D)** $2:1$ **(E)** $4:1$

   Answer (B): The volume of a cylinder with radius r and height h is $V = \pi r^2 h$. Alex's cans have radius 3 cm and height 12 cm, so the volume of one of his cans is $\pi \cdot 3^2 \cdot 12 = 108\pi$ cm^3. Felicia's cans have radius 6 cm and height 6 cm, so the volume of one of her cans is $\pi \cdot 6^2 \cdot 6 = 216\pi$ cm^3. Thus the ratio of the volume of one of Alex's cans to the volume of one of Felicia's cans is $108\pi : 216\pi = 1:2$.

   OR

   Because the volume of a cylindrical can depends upon the height and the square of the radius, when the height is doubled, the volume is multiplied by 2, and when the radius is divided by 2, the volume is divided by 4. Thus doubling the height and halving the radius results in Alex's cans having a volume that is one-half the volume of Felicia's cans. The requested ratio is therefore $1:2$.

10. The diagram shows the number of students at soccer practice each weekday during last week. After computing the mean and median values, Coach discovers that there were actually 21 participants on Wednesday. Which of the following statements describes the change in the mean and median after the correction is made?



(A) The mean increases by 1 and the median does not change.
(B) The mean increases by 1 and the median increases by 1.
(C) The mean increases by 1 and the median increases by 5.
(D) The mean increases by 5 and the median increases by 1.
(E) The mean increases by 5 and the median increases by 5.

Answer (B): The original values in increasing order are 16, 16, 20, 22, and 26, and thus the median (middle) value is 20. After replacing one of the 16s with 21, the new values are 16, 20, 21, 22, and 26, with a median of 21. Therefore the median value increases by 1.

An increase of 5 in the number of participants on one day is equivalent to an increase of 1 participant per day. Therefore the mean (average) value increases by 1. (The mean values could also be directly computed. The mean is 20 for the original set of values and 21 after the mistake has been corrected.)

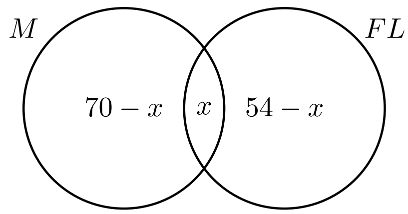
11. The eighth grade class at Lincoln Middle School has 93 students. Each student takes a math class or a foreign language class or both. There are 70 eighth graders taking a math class, and there are 54 eighth graders taking a foreign language class. How many eighth graders take *only* a math class and *not* a foreign language class?

(A) 16 **(B)** 23 **(C)** 31 **(D)** 39 **(E)** 70

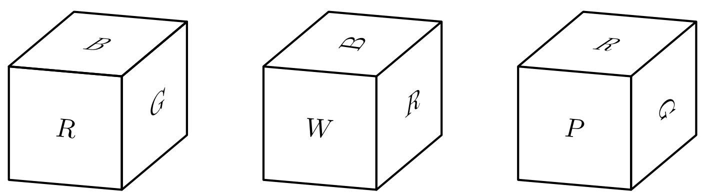
Answer (D): The total enrollment in math classes and foreign language classes is $70 + 54 = 124$ students. There are 93 students, so there must be $124 - 93 = 31$ students who are counted twice because they are taking both a math class and a foreign language class. There are therefore $70 - 31 = 39$ students taking only a math class and not a foreign language class.

OR

In the Venn diagram below, the circle on the left represents the students taking a math class and the circle on the right represents the students taking a foreign language class. Let x represent the number of students taking both a math class and a foreign language class. Then $70 - x$ students take only a math class and $54 - x$ students take only a foreign language class. Because $(70 - x) + x + (54 - x) = 93$, it follows that $x = 31$. There are therefore $70 - 31 = 39$ students who take only a math class and not a foreign language class.



12. The faces of a cube are painted in six different colors: red (R), white (W), green (G), brown (B), aqua (A), and purple (P). Three views of the cube are shown below. What is the color of the face opposite the aqua face?

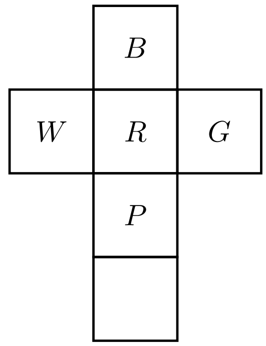


(A) red **(B)** white **(C)** green **(D)** brown **(E)** purple

Answer (A): The second view is a 90° rotation of the first view on the axis through the brown face, implying that the green and white faces are opposite each other. The third view is a 90° rotation of the first view on the axis through the green and white faces. Hence the purple face is opposite the brown face. Then the only face not seen is the aqua face, the back face in the first view and the bottom face in the third view. Thus the red face is opposite the aqua face.

OR

Using the information from the diagrams in the problem, a net for the cube is shown in the following figure. The only face in the net not labeled must be aqua. Refolding the net to make a cube, the red face will be opposite the aqua face.



OR

Using the information from the diagrams in the problem, there are four colors, brown, green, white, and purple, adjacent to the red face. The remaining color, aqua, must be opposite the red face.

13. A *palindrome* is a number that has the same value when read from left to right or from right to left. (For example 12321 is a palindrome.) Let N be the least three-digit integer which is not a palindrome but which is the sum of three distinct two-digit palindromes. What is the sum of the digits of N?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Answer (A): The two-digit palindromes are $11, 22, 33, \ldots, 99$, all of which are multiples of 11. A sum of three of these palindromes will also be a multiple of 11. The smallest three-digit multiple of 11 is 110, and 110 can be expressed as the sum of three two-digit palindromes. For example, $110 = 22 + 33 + 55$. The sum of the digits of 110 is $1 + 1 + 0 = 2$.

Note: Javier Cilleruelo, Florian Luca, and Lewis Baxter recently proved that *any* positive integer can be written as the sum of three palindromes. (See https://arxiv.org/pdf/1602.06208.pdf.)

14. Isabella has 6 coupons that can be redeemed for free ice cream cones at Pete's Sweet Treats. In order to make the coupons last, she decides that she will redeem one every 10 days until she has used them all. She knows that Pete's is closed on Sundays, but as she circles the 6 dates on her calendar, she realizes that no circled date falls on a Sunday. On what day of the week does Isabella redeem her first coupon?

(A) Monday **(B)** Tuesday **(C)** Wednesday **(D)** Thursday

(E) Friday

Answer (C): Because 10 days is equal to 1 week plus 3 days, each of Isabella's circled dates after the first date follows the previous one by 3 days of the week. If she redeems her first coupon on Day x, then she will redeem the others, in order, on Days $x+3$, $x+6$, $x+9$ (which is the same day of the week as $x + 2$), $x + 5$, and $x + 8$ (which is the same day of the week as $x + 1$). The only day of the week when she will not redeem a coupon is $x + 4$, so $x + 4$

must represent Sunday. Therefore she redeems her first coupon 4 days prior to Sunday, which is Wednesday.

OR

If a coupon is redeemed on a Sunday, then the days on which she redeems a coupon are, in order: Wednesday, Saturday, Tuesday, Friday, Monday, Thursday, Sunday, with this cycle of seven days repeating (assuming an unlimited number of coupons). Note that Sunday appears every seventh day in this list. The days that Isabella redeems her coupons also follow the order of this list, starting with the day she redeems the first coupon. Because Sunday is not a day she redeems a coupon, she redeems her six coupons on the six days starting with Wednesday.

15. On a beach 50 people are wearing sunglasses and 35 people are wearing caps. Some people are wearing both sunglasses and caps. If one of the people wearing a cap is selected at random, the probability that this person is also wearing sunglasses is $\frac{2}{5}$. If instead, someone wearing sunglasses is selected at random, what is the probability that this person is also wearing a cap?

(A) $\dfrac{14}{85}$ **(B)** $\dfrac{7}{25}$ **(C)** $\dfrac{2}{5}$ **(D)** $\dfrac{4}{7}$ **(E)** $\dfrac{7}{10}$

Answer (B): Because the probability that a cap-wearing person also wears sunglasses is $\frac{2}{5}$, it follows that the number of people wearing both caps and sunglasses is $\frac{2}{5} \cdot 35 = 14$. Thus the probability that a randomly chosen person with sunglasses is also wearing a cap is $\frac{14}{50} = \frac{7}{25}$.

16. Qiang drives 15 miles at an average speed of 30 miles per hour. How many additional miles will he have to drive at 55 miles per hour to average 50 miles per hour for the entire trip?

(A) 45 **(B)** 62 **(C)** 90 **(D)** 110 **(E)** 135

Answer (D): At 30 mph, Qiang travels the first 15 miles in $\frac{15 \text{ miles}}{30 \text{ mph}} = \frac{1}{2}$ hour. If he had driven the first half hour at 50 mph, he would have traveled 25 miles. So he needs to make up the extra $25 - 15 = 10$ miles in the second part of the trip, in which he is driving 5 mph faster than his overall average. He will require $\frac{10 \text{ miles}}{5 \text{ mph}} = 2$ hours to make up the extra 10 miles. In that time, he will travel $55 \cdot 2 = 110$ miles.

OR

Let t represent the time in hours for the second part of the trip. The distance for the first part of the trip is 15 miles, and the distance for the second part of the trip is the average speed, 55 mph, times t. The distance for the entire trip is the average speed, 50 mph, times the total time, $\frac{1}{2} + t$. Thus

$$15 + 55t = 50\left(\frac{1}{2} + t\right)$$

and it follows that $t = 2$ hours. The distance traveled in 2 hours at 55 mph is 110 miles.

17. What is the value of the product

$$\left(\frac{1\cdot 3}{2\cdot 2}\right)\left(\frac{2\cdot 4}{3\cdot 3}\right)\left(\frac{3\cdot 5}{4\cdot 4}\right)\cdots\left(\frac{97\cdot 99}{98\cdot 98}\right)\left(\frac{98\cdot 100}{99\cdot 99}\right)?$$

(A) $\frac{1}{2}$ **(B)** $\frac{50}{99}$ **(C)** $\frac{9800}{9801}$ **(D)** $\frac{100}{99}$ **(E)** 50

Answer (B): The product may be rewritten as

$$\left(\frac{1}{2}\right)\left(\frac{3\cdot 2}{2\cdot 3}\right)\left(\frac{4\cdot 3}{3\cdot 4}\right)\left(\frac{5\cdot 4}{4\cdot 5}\right)\cdots\left(\frac{99\cdot 98}{98\cdot 99}\right)\left(\frac{100}{99}\right)=\frac{1}{2}\cdot\frac{100}{99}=\frac{50}{99}.$$

18. The faces on each of two fair dice are numbered 1, 2, 3, 5, 7, and 8. When the two dice are tossed, what is the probability that their sum will be an even number?

(A) $\frac{4}{9}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{5}{9}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{2}{3}$

Answer (C): The table below shows that 20 of the 36 sums will be even.

+	1	2	3	5	7	8
1	2	3	4	6	8	9
2	3	4	5	7	9	10
3	4	5	6	8	10	11
5	6	7	8	10	12	13
7	8	9	10	12	14	15
8	9	10	11	13	15	16

Therefore the probability that the sum will be even is $\frac{20}{36}=\frac{5}{9}$.

OR

The sum is even provided both dice are even or both dice are odd. Because there is a $\frac{1}{3}$ probability of one die being even, there is a $\frac{1}{9}$ probability that both will be even. There is a $\frac{2}{3}$ probability that one will be odd and thus a $\frac{4}{9}$ probability that both will be odd. Thus the overall probability that the sum is even is $\frac{1}{9}+\frac{4}{9}=\frac{5}{9}$.

19. In a tournament there are six teams that play each other twice. A team earns 3 points for a win, 1 point for a draw, and 0 points for a loss. After all the games have been played it turns out that the top three teams earned the same number of total points. What is the greatest possible number of total points for each of the top three teams?

(A) 22 **(B)** 23 **(C)** 24 **(D)** 26 **(E)** 30

Answer (C): The top three teams played six games against one another in which at most $6\cdot 3=18$ points were awarded, and they played 18 games against the other three teams in which at most $18 \cdot 3 = 54$ points were awarded. Therefore the sum of the scores of the top three teams was at most 72, and thus the total score of each of the top teams was at most 24. This score is indeed obtained when between each pair of top teams, one game is won and one game is lost, and each top team wins both games against each of the other three teams.

20. How many different real numbers x satisfy the equation

$$\left(x^2 - 5\right)^2 = 16\,?$$

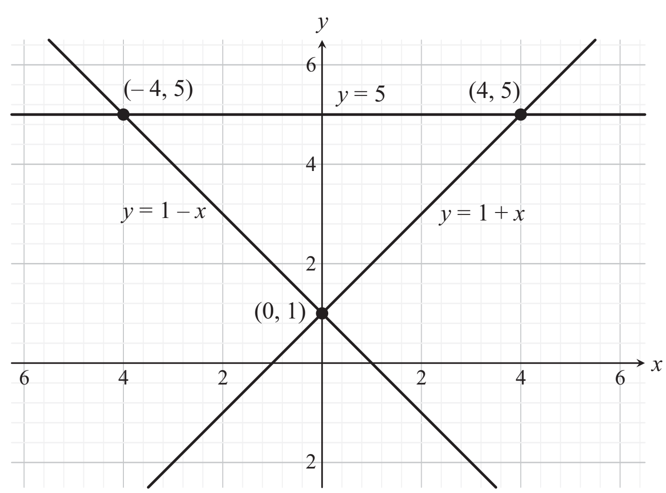
(A) 0 **(B)** 1 **(C)** 2 **(D)** 4 **(E)** 8

Answer (D): If x is a real number satisfying $\left(x^2-5\right)^2 = 16$ then either $x^2 - 5 = 4$ or $x^2 - 5 = -4$. There are two numbers, 3 and -3, that satisfy the first equation and two additional numbers, 1 and -1, that satisfy the second equation. Thus there are four distinct solutions in all.

21. What is the area of the triangle formed by the lines $y = 5$, $y = 1 + x$, and $y = 1 - x$?

(A) 4 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 16

Answer (E): These three lines enclose a triangle whose vertices are $(0, 1)$, $(4, 5)$, and $(-4, 5)$. The base of this triangle on the line $y = 5$ has length 8, and its altitude on the y-axis has length 4. Thus the area of the triangle is $\frac{1}{2} \cdot 8 \cdot 4 = 16$.



22. A store increased the original price of a shirt by a certain percent and then decreased the new price by the same percent. Given that the resulting price was 84% of the original price, by what percent was the price increased and decreased?

(A) 16 **(B)** 20 **(C)** 28 **(D)** 36 **(E)** 40

Answer (E): Suppose the original price of the shirt was 100 dollars and the price was increased by x percent, producing a new price of $100 + x$ dollars. Then decreasing the new price by x percent is equivalent to multiplying $100 + x$ by $(100 - x)$ percent. The resulting price was 84 dollars. Thus

$$\begin{aligned}(100 + x) \cdot \frac{100 - x}{100} &= 84 \\ 100^2 - x^2 &= 8400 \\ x^2 &= 1600 \\ x &= 40\end{aligned}$$

Therefore the percent of increase and decrease was 40%.

23. After Euclid High School's last basketball game, it was determined that $\frac{1}{4}$ of the team's points were scored by Alexa and $\frac{2}{7}$ were scored by Brittany. Chelsea scored 15 points. None of the other 7 team members scored more than 2 points. What was the total number of points scored by the other 7 team members?

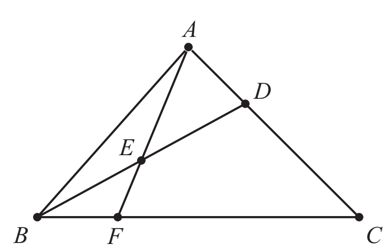
(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

Answer (B): The total number of points scored by players other than Alexa and Brittany was at least 15 and at most $15 + 7 \cdot 2 = 29$. The fraction of the team's points scored by those players was $1 - \frac{1}{4} - \frac{2}{7} = \frac{13}{28}$, so the number of points that they scored was a multiple of 13. The only multiple of 13 between 15 and 29 is 26, so the number of points scored by the 7 players other than Alexa, Brittany, and Chelsea was $26 - 15 = 11$.

OR

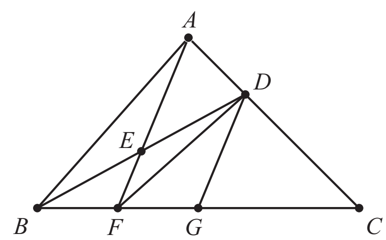
The total number of points scored by the team was a multiple of both 4 and 7 and thus was a multiple of 28. If the total was 28, then Alexa scored 7 and Brittany scored 8, and so Chelsea could not have scored 15. If the total was $28n$, where $n \geq 3$, then Alexa scored $7n$ and Brittany scored $8n$. With Chelsea scoring 15, this leaves $13n - 15 \geq 24$ points for the remaining 7 team members. But then at least one of them scored more than 2 points. So the only possible score was $28 \cdot 2 = 56$. Then Alexa scored 14, Brittany scored 16, Chelsea scored 15, and the remaining 7 team members scored a total of 11 points.

24. In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?



(A) 24 **(B)** 30 **(C)** 32 **(D)** 36 **(E)** 40

Answer (B):



Let G be the point on side BC such that $\overline{DG}$ is parallel to $\overline{AF}$. Then $BF = FG$ because $\triangle BEF$ and $\triangle BDG$ are similar and $BE = ED$. Also $2FG = GC$ because $\triangle ACF$ and $\triangle DCG$ are similar and $2AD = DC$. Thus $BC = BF + FG + GC = 4BF$. Let $[XYZ]$ denote the area of a triangle XYZ. Then

$$[BDC] = \frac{2}{3}[BAC] = 240$$

because $DC = 2AD$ and the triangles have the same altitude to line AC. Draw segment $\overline{FD}$. Similarly, $[BDF] = \frac{1}{4}[BDC] = 60$. Therefore $[BEF] = \frac{1}{2}[BDF] = 30$.

OR

Because triangles ABD and DBC share the same altitude to line AC and $AD : DC = 1 : 2$, the ratio of the areas $[ABD] : [DBC]$ also is $1 : 2$. Draw segment $\overline{FD}$. Similarly the ratio $[AFD] : [DFC]$ is $1 : 2$.

Let x represent the area of $\triangle EBF$. Then because $BE = ED$, the areas $[ABE]$ and $[AED]$ are equal and $[EBF] = [DEF] = x$.

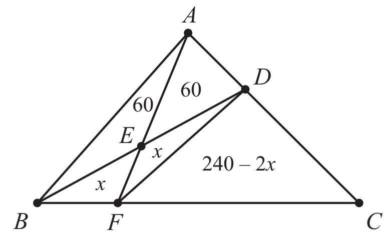
Given that the area of $\triangle ABC$ is 360, we can calculate the following areas:

$$[ABD] = \frac{1}{3}[ABC] = \frac{1}{3}(360) = 120$$
$$[DBC] = \frac{2}{3}[ABC] = \frac{2}{3}(360) = 240$$
$$[ABE] = [AED] = \frac{1}{2}[ABD] = \frac{1}{2}(120) = 60.$$

Then using the ratio of the areas of $\triangle AFD$ and $\triangle DFC$ we can solve for x.

$$\frac{[AFD]}{[DFC]} = \frac{1}{2}$$
$$\frac{[AED] + [DEF]}{[DBC] - [DBF]} = \frac{1}{2}$$
$$\frac{60 + x}{240 - 2x} = \frac{1}{2}$$
$$120 + 2x = 240 - 2x$$
$$4x = 120$$
$$x = 30.$$

Therefore the area of $\triangle EBF$ is 30.



25. Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples?

(A) 105 **(B)** 114 **(C)** 190 **(D)** 210 **(E)** 380

Answer (C): Make a table of possibilities focusing on Alice.

Alice	Becky	Chris
20	2	2
19	3	2
19	2	3
18	4	2
18	3	3
18	2	4
$\vdots$	$\vdots$	$\vdots$
2	20	2
$\vdots$	$\vdots$	$\vdots$
2	2	20

There are $1 + 2 + \cdots + 18 + 19 = (19)(20)/2 = 190$ ways for Alice to share the apples.

OR

Apply the sticks and stones counting method. Imagine 24 stones in a row to represent the apples and 23 sticks separating them.

$$\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc|\bigcirc$$

To divide the apples into three groups, two of the sticks must be selected. The sticks at either end cannot be selected because each person must receive at least two apples. The number of ways to choose 2 of the remaining 21 sticks, denoted by ${}_{21}C_2$, is $\frac{21 \cdot 20}{2} = 210$. We also cannot select two adjacent sticks for the same reason. With the first and last sticks excluded, there are 20 pairs of adjacent sticks. Therefore, there are ${}_{21}C_2 - 20 = 210 - 20 = 190$ ways to select the sticks and thereby distribute the apples.

OR

Imagine first giving each of the girls two apples. Now there are 18 apples remaining. Each way to distribute these apples among the three girls can be modeled by an arrangement of 18 stones and 2 sticks. For example,

○ ○ ○ ○ ○ ○ ○ | | ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○

corresponds to giving 7 more apples to Alice, none to Becky, and 11 more to Chris. The number of arrangements of 18 stones and 2 sticks is ${}_{20}C_2 = \frac{20!}{18!\,2!} = 190.$

Problems and solutions were contributed by Risto Atanasov, Bela Bajnok, Steve Blasberg, Thomas Butts, Silva Chang, Barbara Currier, Steven Davis, Steven Dunbar, Marta Eso, John Frohliger, Thomas Hagedorn, Joe Kennedy, Norbert Kuenzi, Andrea Rothbart, Qiang Shi, Jeganathan Sriskandarajah, Zsuzsanna Szaniszlo, David Wells, and Carl Yerger.

MAA Partner Organizations

We acknowledge the generosity of the following organizations in supporting the MAA AMC and Invitational Competitions:

Patron's Circle

Akamai Foundation
TBL Foundation

Innovator's Circle

The D. E. Shaw Group
Two Sigma

Winner's Circle

MathWorks
Tudor Investment Corporation

Achiever's Circle

Art of Problem Solving
Jane Street Capital

Sustainer's Circle

Ansatz Capital
Army Educational Outreach Program

Collaborator's Circle

American Mathematical Society
American Statistical Association
Casualty Actuarial Society
Mu Alpha Theta
Society for Industrial and Applied Mathematics