MAA AMC

American Mathematics Competitions

MAA American Mathematics Competitions

23rd Annual

AMC 10 B

FALL

Tuesday, November 16, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR COMPETITION MANAGER TELLS YOU TO BEGIN.
2. This is a 25-question multiple-choice competition. For each question, only one answer choice is correct.
3. Mark your answer to each problem on the answer sheet with a #2 pencil. Check blackened answers for accuracy and erase errors completely. Only answers that are properly marked on the answer sheet will be scored.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, blank graph paper, rulers, compasses, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the competition will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the competition, your competition manager will ask you to record your name on the answer sheet.
8. You will have 75 minutes to complete the competition once your competition manager tells you to begin.
9. When you finish the competition, sign your name in the space provided on the answer sheet and complete the demographic information questions on the back of the answer sheet.

The MAA AMC office reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

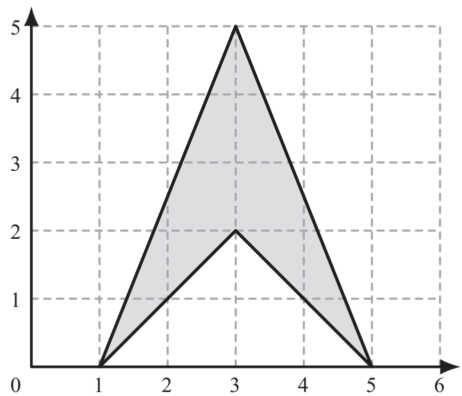
The publication, reproduction, or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

Students who score well on this AMC 10 will be invited to take the 40th annual American Invitational Mathematics Examination (AIME) on Tuesday, February 8, 2022, or Wednesday, February 16, 2022. More details about the AIME are on the back page of this test booklet.

1. What is the value of $1234 + 2341 + 3412 + 4123$?

   (A) 10,000 **(B)** 10,010 **(C)** 10,110 **(D)** 11,000 **(E)** 11,110

2. What is the area of the shaded figure shown below?



   (A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

3. The expression $\frac{2021}{2020} - \frac{2020}{2021}$ is equal to the fraction $\frac{p}{q}$ in which p and q are positive integers whose greatest common divisor is 1. What is p ?

   (A) 1 **(B)** 9 **(C)** 2020 **(D)** 2021 **(E)** 4041

4. At noon on a certain day, Minneapolis is N degrees warmer than St. Louis. At 4:00 the temperature in Minneapolis has fallen by 5 degrees while the temperature in St. Louis has risen by 3 degrees, at which time the temperatures in the two cities differ by 2 degrees. What is the product of all possible values of N ?

   (A) 10 **(B)** 30 **(C)** 60 **(D)** 100 **(E)** 120

5. Let $n = 8^{2022}$. Which of the following is equal to $\frac{n}{4}$?

   (A) 4^{1010} **(B)** 2^{2022} **(C)** 8^{2018} **(D)** 4^{3031} **(E)** 4^{3032}

6. The least positive integer with exactly 2021 distinct positive divisors can be written in the form $m \cdot 6^k$, where m and k are integers and 6 is not a divisor of m. What is $m + k$?

   (A) 47 **(B)** 58 **(C)** 59 **(D)** 88 **(E)** 90

7. Call a fraction $\frac{a}{b}$, not necessarily in simplest form, *special* if a and b are positive integers whose sum is 15. How many distinct integers can be written as the sum of two, not necessarily different, special fractions?

   (A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

8. The greatest prime number that is a divisor of 16,384 is 2 because $16{,}384 = 2^{14}$. What is the sum of the digits of the greatest prime number that is a divisor of 16,383 ?

   (A) 3 **(B)** 7 **(C)** 10 **(D)** 16 **(E)** 22

9. The knights in a certain kingdom come in two colors: $\frac{2}{7}$ of them are red, and the rest are blue. Furthermore, $\frac{1}{6}$ of the knights are magical, and the fraction of red knights who are magical is 2 times the fraction of blue knights who are magical. What fraction of red knights are magical?

   (A) $\dfrac{2}{9}$ **(B)** $\dfrac{3}{13}$ **(C)** $\dfrac{7}{27}$ **(D)** $\dfrac{2}{7}$ **(E)** $\dfrac{1}{3}$

10. Forty slips of paper numbered 1 to 40 are placed in a hat. Alice and Bob each draw one number from the hat without replacement, keeping their numbers hidden from each other. Alice says, “I can’t tell who has the larger number.” Then Bob says, “I know who has the larger number.” Alice says, “You do? Is your number prime?” Bob replies, “Yes.” Alice says, “In that case, if I multiply your number by 100 and add my number, the result is a perfect square.” What is the sum of the two numbers drawn from the hat?

    (A) 27 **(B)** 37 **(C)** 47 **(D)** 57 **(E)** 67

11. A regular hexagon of side length 1 is inscribed in a circle. Each minor arc of the circle determined by a side of the hexagon is reflected over that side. What is the area of the region bounded by these 6 reflected arcs?

    (A) $\dfrac{5\sqrt{3}}{2} - \pi$ **(B)** $3\sqrt{3} - \pi$ **(C)** $4\sqrt{3} - \dfrac{3\pi}{2}$ **(D)** $\pi - \dfrac{\sqrt{3}}{2}$

    (E) $\dfrac{\pi + \sqrt{3}}{2}$

12. Which of the following conditions is sufficient to guarantee that integers x, y, and z satisfy the equation

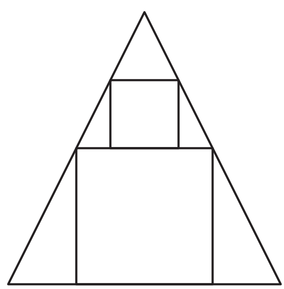
$$x(x-y)+y(y-z)+z(z-x)=1\,?$$

(A) $x>y$ and $y=z$ **(B)** $x=y-1$ and $y=z-1$

(C) $x=z+1$ and $y=x+1$ **(D)** $x=z$ and $y-1=x$

(E) $x+y+z=1$

13. A square with side length 3 is inscribed in an isosceles triangle with one side of the square along the base of the triangle. A square with side length 2 has two vertices on the other square and the other two on sides of the triangle, as shown. What is the area of the triangle?

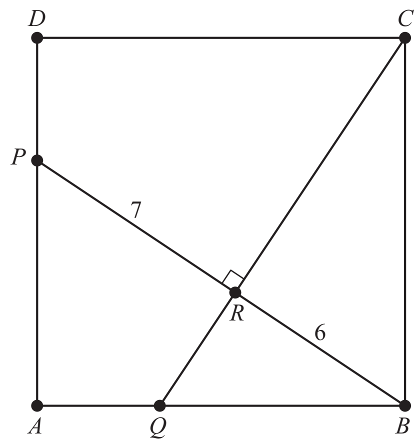


(A) $19\frac{1}{4}$ **(B)** $20\frac{1}{4}$ **(C)** $21\frac{3}{4}$ **(D)** $22\frac{1}{2}$ **(E)** $23\frac{3}{4}$

14. Una rolls 6 standard 6-sided dice simultaneously and calculates the product of the 6 numbers obtained. What is the probability that the product is divisible by 4 ?

(A) $\frac{3}{4}$ **(B)** $\frac{57}{64}$ **(C)** $\frac{59}{64}$ **(D)** $\frac{187}{192}$ **(E)** $\frac{63}{64}$

15. In square $ABCD$, points P and Q lie on $\overline{AD}$ and $\overline{AB}$, respectively. Segments $\overline{BP}$ and $\overline{CQ}$ intersect at right angles at R, with $BR = 6$ and $PR = 7$. What is the area of the square?



(A) 85 **(B)** 93 **(C)** 100 **(D)** 117 **(E)** 125

16. Five balls are arranged around a circle. Chris chooses two adjacent balls at random and interchanges them. Then Silva does the same, with her choice of adjacent balls to interchange being independent of Chris's. What is the expected number of balls that occupy their original positions after these two successive transpositions?

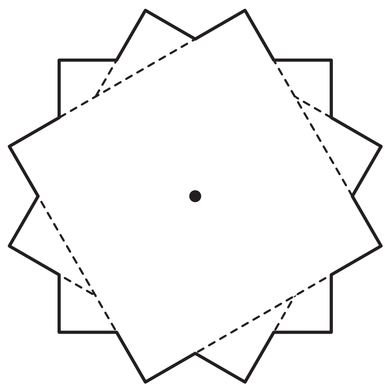
(A) 1.6 **(B)** 1.8 **(C)** 2.0 **(D)** 2.2 **(E)** 2.4

17. Distinct lines ℓ and m lie in the xy-plane. They intersect at the origin. Point $P(-1, 4)$ is reflected about line ℓ to point P', and then P' is reflected about line m to point P''. The equation of line ℓ is $5x - y = 0$, and the coordinates of P'' are $(4, 1)$. What is the equation of line m ?

(A) $5x + 2y = 0$ **(B)** $3x + 2y = 0$ **(C)** $x - 3y = 0$

(D) $2x - 3y = 0$ **(E)** $5x - 3y = 0$

18. Three identical square sheets of paper each with side length 6 are stacked on top of each other. The middle sheet is rotated clockwise 30° about its center and the top sheet is rotated clockwise 60° about its center, resulting in the 24-sided polygon shown in the figure below. The area of this polygon can be expressed in the form $a - b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. What is $a + b + c$?



(A) 75 **(B)** 93 **(C)** 96 **(D)** 129 **(E)** 147

19. Let N be the positive integer $7777\ldots777$, a 313-digit number where each digit is a 7. Let $f(r)$ be the leading digit of the rth root of N. What is $f(2) + f(3) + f(4) + f(5) + f(6)$?

(A) 8 **(B)** 9 **(C)** 11 **(D)** 22 **(E)** 29

20. In a particular game, each of 4 players rolls a standard 6-sided die. The winner is the player who rolls the highest number. If there is a tie for the highest roll, those involved in the tie will roll again and this process will continue until one player wins. Hugo is one of the players in this game. What is the probability that Hugo's first roll was a 5, given that he won the game?

(A) $\dfrac{61}{216}$ **(B)** $\dfrac{367}{1296}$ **(C)** $\dfrac{41}{144}$ **(D)** $\dfrac{185}{648}$ **(E)** $\dfrac{11}{36}$

21. Regular polygons with 5, 6, 7, and 8 sides are inscribed in the same circle. No two of the polygons share a vertex, and no three of their sides intersect at a common point. At how many points inside the circle do two of their sides intersect?

(A) 52 **(B)** 56 **(C)** 60 **(D)** 64 **(E)** 68

22. For each integer $n \geq 2$, let S_n be the sum of all products jk, where j and k are integers and $1 \leq j < k \leq n$. What is the sum of the 10 least values of n such that S_n is divisible by 3 ?

(A) 196 **(B)** 197 **(C)** 198 **(D)** 199 **(E)** 200

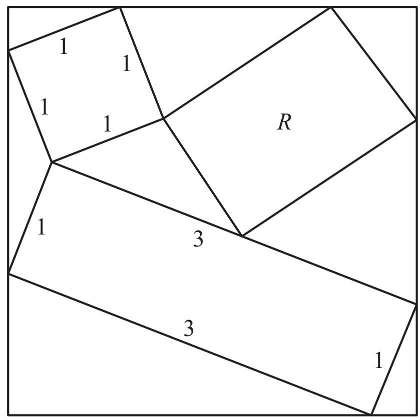
23. Each of the 5 sides and the 5 diagonals of a regular pentagon are randomly and independently colored red or blue with equal probability. What is the probability that there will be a triangle whose vertices are among the vertices of the pentagon such that all of its sides have the same color?

(A) $\frac{2}{3}$ **(B)** $\frac{105}{128}$ **(C)** $\frac{125}{128}$ **(D)** $\frac{253}{256}$ **(E)** 1

24. A cube is constructed from 4 white unit cubes and 4 blue unit cubes. How many different ways are there to construct the $2 \times 2 \times 2$ cube using these smaller cubes? (Two constructions are considered the same if one can be rotated to match the other.)

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

25. A rectangle with side lengths 1 and 3, a square with side length 1, and a rectangle R are inscribed inside a larger square as shown. The sum of all possible values for the area of R can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?



(A) 14 **(B)** 23 **(C)** 46 **(D)** 59 **(E)** 67

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Scores and official competition solutions will be sent to your competition manager, who can share that information with you.

For more information about the MAA American Mathematics Competitions program and our other competitions, please visit maa.org/amc.

Questions and comments about this competition should be sent to

amcinfo@maa.org

or

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P.O. Box 471
Annapolis Junction, MD 20701.

The problems and solutions for this AMC 10 B were prepared by the MAA AMC 10/12 Editorial Board under the direction of Azar Khosravani and Carl Yerger, co-Editors-in-Chief.

MAA Partner Organizations

We acknowledge the generosity of the following organizations in supporting the MAA AMC and Invitational Competitions:

Akamai Foundation

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DO NOT OPEN until Tuesday, November 16, 2021

****Administration on an earlier date will disqualify your school's results.****

- All the information needed to administer this competition is contained in the AMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL BEFORE TUESDAY, NOVEMBER 16, 2021.
- Answer sheets must be returned to the MAA AMC office within 24 hours of the competition administration. Use an overnight or 2-day shipping service, with a tracking number, to guarantee timely arrival of these answer sheets. FedEx, UPS, or USPS overnight are strongly recommended.
- The 40th annual American Invitational Mathematics Exam will be held on Tuesday, February 8, 2022, with an alternate date on Wednesday, February 16, 2022. It is a 15-question, 3-hour, integer-answer competition. Students who achieve a high score on the AMC 10 will be invited to participate. Top-scoring students on the AMC 10/12 and AIME will be selected to take the USA (Junior) Mathematical Olympiad.
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