MAA AMC
American Mathematics Competitions

SPRING

# Official Solutions

MAA American Mathematics Competitions

22nd Annual

# AMC 10 A

**Thursday, February 4, 2021**

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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The problems and solutions for this AMC 10 A were prepared by the
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Azar Khosravani and Carl Yerger, co-Editors-in-Chief.

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1. What is the value of
$$\left(2^2-2\right)-\left(3^2-3\right)+\left(4^2-4\right)\,?$$

**(A)** 1 **(B)** 2 **(C)** 5 **(D)** 8 **(E)** 12

**Answer (D):**
$$\left(2^2-2\right)-\left(3^2-3\right)+\left(4^2-4\right)=2-6+12=8.$$

2. Portia's high school has 3 times as many students as Lara's high school. The two high schools have a total of 2600 students. How many students does Portia's high school have?

**(A)** 600 **(B)** 650 **(C)** 1950 **(D)** 2000 **(E)** 2050

**Answer (C):** One fourth of the 2600 students are in Lara's high school, and three fourths are in Portia's high school. Therefore $\frac{3}{4}\cdot 2600 = 1950$ students are in Portia's high school.

3. The sum of two natural numbers is 17,402. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

**(A)** 10,272 **(B)** 11,700 **(C)** 13,362 **(D)** 14,238 **(E)** 15,426

**Answer (D):** Let $n$ be the number that is known to be a multiple of 10. Then the other number is $\frac{n}{10}$. The sum condition says that $n+\frac{n}{10}=17{,}402$, which can be rewritten as $\frac{11n}{10}=17{,}402$. Multiplying both sides by $\frac{10}{11}$ gives $n=15{,}820$. The requested difference is $15{,}820-1{,}582=14{,}238$.

**OR**

Let $s$ be the lesser number. Then the greater number is $10s$ and the sum of the two numbers is $11s=17{,}402$. This gives $s=1{,}582$, and the difference of the two numbers is $9s=14{,}238$.

4. A cart rolls down a hill, traveling 5 inches the first second and accelerating so that during each successive 1-second time interval, it travels 7 inches more than during the previous 1-second interval. The cart takes 30 seconds to reach the bottom of the hill. How far, in inches, does it travel?

**(A)** 215 **(B)** 360 **(C)** 2992 **(D)** 3195 **(E)** 3242

**Answer (D):** The distance traveled during the 1st, 2nd, 3rd, …, 30th second is $5, 5+7, 5+2\cdot7, \ldots, 5+29\cdot7$, respectively. Therefore the total distance traveled is
$$\begin{aligned}5+(5+7)+(5+2\cdot7)+\cdots+(5+29\cdot7)&\\ &=5\cdot30+(7+2\cdot7+\cdots+29\cdot7)\\ &=150+7\cdot(1+2+\cdots+29)\\ &=150+7\cdot\frac{29\cdot30}{2}\\ &=3195.\end{aligned}$$

5. The quiz scores of a class with $k>12$ students have a mean of 8. The mean of a collection of 12 of these quiz scores is 14. What is the mean of the remaining quiz scores in terms of $k$ ?

**(A)** $\dfrac{14-8}{k-12}$ **(B)** $\dfrac{8k-168}{k-12}$ **(C)** $\dfrac{14}{12}-\dfrac{8}{k}$ **(D)** $\dfrac{14(k-12)}{k^2}$
**(E)** $\dfrac{14(k-12)}{8k}$

**Answer (B):** The total number of points earned on all the quizzes is $8k$, and the number of points earned on the 12 quizzes is $12 \cdot 14 = 168$. Therefore the number of points earned on the $k - 12$ remaining quizzes is $8k - 168$. The mean of the scores of the remaining quizzes is thus

$$\frac{8k - 168}{k - 12}.$$

6. Chantal and Jean start hiking from a trailhead toward a fire tower. Jean is wearing a heavy backpack and walks slower. Chantal starts walking at 4 miles per hour. Halfway to the tower, the trail becomes really steep, and Chantal slows down to 2 miles per hour. After reaching the tower, she immediately turns around and descends the steep part of the trail at 3 miles per hour. She meets Jean at the halfway point. What was Jean's average speed, in miles per hour, until they meet?

**(A)** $\frac{12}{13}$ **(B)** 1 **(C)** $\frac{13}{12}$ **(D)** $\frac{24}{13}$ **(E)** 2

**Answer (A):** Recall that distance equals rate times time, so time equals distance divided by rate. Let $d$ denote the distance to the fire tower, in miles. The elapsed time, in hours, in the period described is $\frac{d}{2} \div 4 = \frac{d}{8}$ for Chantal to reach the halfway point, plus $\frac{d}{2} \div 2 = \frac{d}{4}$ for her to hike from the halfway point to the tower, plus $\frac{d}{2} \div 3 = \frac{d}{6}$ for her to hike back from the tower to the halfway point. The total time elapsed is $\frac{d}{8} + \frac{d}{4} + \frac{d}{6} = \frac{13}{24}d$. Because Jean hikes distance $\frac{d}{2}$ in this time, her rate is $\frac{d}{2} \div \left(\frac{13}{24}d\right) = \frac{12}{13}$.

7. Tom has a collection of 13 snakes, 4 of which are purple and 5 of which are happy. He observes that

- all of his happy snakes can add,
- none of his purple snakes can subtract, and
- all of his snakes that can't subtract also can't add.

Which of these conclusions can be drawn about Tom's snakes?

**(A)** Purple snakes can add.

**(B)** Purple snakes are happy.

**(C)** Snakes that can add are purple.

**(D)** Happy snakes are not purple.

**(E)** Happy snakes can't subtract.

**Answer (D):** To see that choices **(A)**, **(B)**, **(C)**, and **(E)** do not follow from the given informtion, consider the following two snakes that may be part of Tom's collection. One snake is happy but not purple and can both add and subtract. The second is purple but not happy and can neither add nor subtract. Then each of the three bulleted statements is true, but each of these choices is false.

To show that answer choice **(D)** is correct, first observe that the third bulleted statement is equivalent to "Snakes that can add also can subtract." The second bulleted statement is equivalent to "Snakes that can subtract are not purple." The three bulleted statements combined then lead to the conclusion "Happy snakes are not purple."

**OR**

Let $H$, $P$, $A$, and $S$ denote the statements that a snake is happy, is purple, can add, and can subtract, respectively. Let $\to$ denote "implies" and $\sim$ denote "not". Recall that an implication $X \to Y$ is logically equivalent to its contrapositive

$$(\sim Y) \to (\sim X).$$

Then the three bulleted statements can be exactly summarized as

$$H \to A \to S \to (\sim P).$$

Choice **(D)**, which is

$$H \to (\sim P),$$

follows from the transitivity of the "implies" relation. However, choices **(A)**, which is $P \to A$; **(B)**, which is $P \to H$; **(C)**, which is $A \to P$; and **(E)**, which is $H \to (\sim S)$, do not follow from those three implications.

8. When a student multiplied the number 66 by the repeating decimal,

$$\underline{1}.\underline{a}\,\underline{b}\,\underline{a}\,\underline{b}\ldots = \underline{1}.\overline{\underline{a}\,\underline{b}},$$

where $a$ and $b$ are digits, he did not notice the notation and just multiplied 66 times $\underline{1}.\underline{a}\,\underline{b}$. Later he found that his answer is 0.5 less than the correct answer. What is the 2-digit integer $\underline{a}\,\underline{b}$ ?

**(A)** 15 **(B)** 30 **(C)** 45 **(D)** 60 **(E)** 75

**Answer (E):** The given condition is

$$66 \cdot \underline{1}.\underline{a}\,\underline{b} = 66 \cdot \underline{1}.\overline{\underline{a}\,\underline{b}} - 0.5,$$

which is equivalent to

$$0.5 = 66 \cdot \underline{1}.\overline{\underline{a}\,\underline{b}} - 66 \cdot \underline{1}.\underline{a}\,\underline{b} = 66(\underline{1}.\overline{\underline{a}\,\underline{b}} - \underline{1}.\underline{a}\,\underline{b}) = 66 \cdot \underline{0}.\underline{0}\,\underline{0}\,\overline{\underline{a}\,\underline{b}}.$$

Because

$$\underline{0}.\overline{\underline{a}\,\underline{b}} = \frac{\underline{a}\,\underline{b}}{99},$$

this is equivalent to

$$\frac{1}{2} = 66 \cdot \frac{\underline{a}\,\underline{b}}{9900},$$

from which $\underline{a}\,\underline{b} = 75$.

9. What is the least possible value of $(xy-1)^2 + (x+y)^2$ for real numbers $x$ and $y$ ?

**(A)** 0 **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 2

**Answer (D):** Because

$$(xy-1)^2 + (x+y)^2 = x^2y^2 + x^2 + y^2 + 1 = \left(x^2+1\right)\left(y^2+1\right)$$

and both factors are at least 1, the least possible value of the expression is 1. It occurs when $x = 0$ and $y = 0$.

10. Which of the following is equivalent to

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})\,?$$

**(A)** $3^{127}+2^{127}$ **(B)** $3^{127}+2^{127}+2\cdot 3^{63}+3\cdot 2^{63}$ **(C)** $3^{128}-2^{128}$

**(D)** $3^{128}+2^{128}$ **(E)** $5^{127}$

**Answer (C):** The product telescopes as follows:

$$\begin{aligned}
&(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)\cdots(2^{64}+3^{64})\\
&\quad= (3-2)(3+2)(3^2+2^2)(3^4+2^4)(3^8+2^8)\cdots(3^{64}+2^{64})\\
&\quad= (3^2-2^2)(3^2+2^2)(3^4+2^4)(3^8+2^8)\cdots(3^{64}+2^{64})\\
&\quad= (3^4-2^4)(3^4+2^4)(3^8+2^8)\cdots(3^{64}+2^{64})\\
&\quad\ \vdots\\
&\quad= (3^{64}-2^{64})(3^{64}+2^{64})\\
&\quad= 3^{128}-2^{128}.
\end{aligned}$$

11. For which of the following integers $b$ is the base-$b$ number $2021_b - 221_b$ not divisible by 3 ?

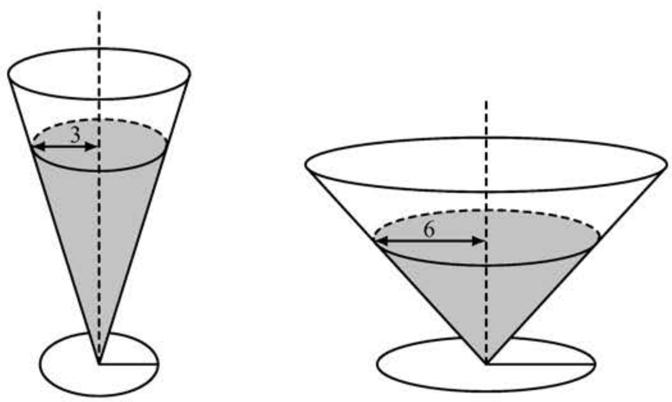
**(A)** 3 **(B)** 4 **(C)** 6 **(D)** 7 **(E)** 8

**Answer (E):** Note that

$$2021_b - 221_b = 2b^3 + 2b + 1 - \big(2b^2 + 2b + 1\big) = 2b^3 - 2b^2 = 2b^2(b-1).$$

This number is divisible by 3 if and only if either $b$ or $b-1$ is divisible by 3. Because neither 8 nor $8-1$ is divisible by 3, the base-eight number $2021_{\text{eight}} - 221_{\text{eight}}$ is not divisible by 3. For all the other given choices for $b$, either $b$ or $b-1$ is divisible by 3.

12. Two right circular cones with vertices facing down as shown in the figure below contain the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into each cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?




**(A)** 1 : 1 **(B)** 47 : 43 **(C)** 2 : 1 **(D)** 40 : 13 **(E)** 4 : 1

**Answer (E):** If two cones have the same volume and the radius of the narrow cone is one-half the radius of the wide cone, then because the volume of a cone varies directly with both the square of the radius and the height, it follows that the height of the narrow cone is 4 times the height of the wide cone. The volumes of liquid are the same both before and after the marble is added, so both heights for the narrow cone are 4 times the corresponding heights for the wide cone. Hence the rise of the level in the narrow cone is also 4 times the rise of the level in the wide cone.

**OR**

Let $h_1$ and $h_2$ be the heights of the liquid in the narrow and wide cones, respectively, before the marble is dropped in. The two equal volumes of liquid are

$$\frac{1}{3}\pi \cdot 3^2 \cdot h_1 = \frac{1}{3}\pi \cdot 6^2 \cdot h_2,$$

which implies $h_1 = 4h_2$. When the marble of volume $\frac{4}{3}\pi$ is added, both the heights and the radii of the conical volumes of water change, but the volumes remain equal. Let $R_1$ and $H_1$ be the new radius and height for the new narrow cone, and $R_2$ and $H_2$ be the new radius and height for the new wide cone. Then

$$\frac{1}{3}\pi R_1^2 H_1 = 3\pi h_1 + \frac{4}{3}\pi.$$

By similar triangles, $R_1 = \frac{3H_1}{h_1}$. Substituting and solving for $H_1$ gives

$$H_1 = \sqrt[3]{h_1^3 + \frac{4}{9}h_1^2}.$$

Letting $h_1 = 4h_2$ gives

$$H_1 = \sqrt[3]{64h_2^3 + \frac{64}{9}h_2^2},$$

which is equivalent to

$$H_1 = 4\sqrt[3]{h_2^3 + \frac{1}{9}h_2^2}.$$

Similarly,

$$\frac{1}{3}\pi R_2^2 H_2 = 12\pi h_2 + \frac{4}{3}\pi,$$

and by similar triangles, $R_2 = \frac{6H_2}{h_2}$. Substituting and solving for $H_2$ gives

$$H_2 = \sqrt[3]{h_2^3 + \frac{1}{9}h_2^2}.$$

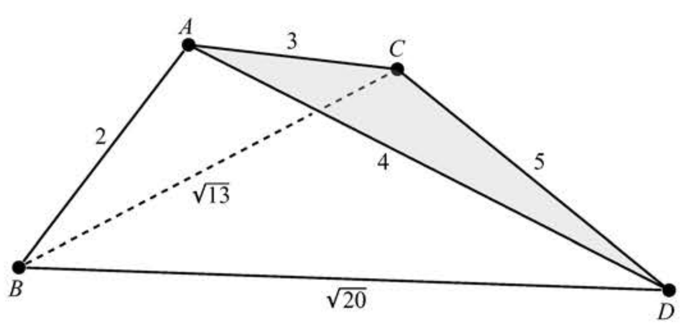
It follows that the ratio of the rise of the water levels is

$$\frac{H_1 - h_1}{H_2 - h_2} = \frac{4\sqrt[3]{h_2^3 + \frac{1}{9}h_2^2} - 4h_2}{\sqrt[3]{h_2^3 + \frac{1}{9}h_2^2} - h_2} = 4.$$

13. What is the volume of tetrahedron $ABCD$ with edge lengths $AB = 2$, $AC = 3$, $AD = 4$, $BC = \sqrt{13}$, $BD = 2\sqrt{5}$, and $CD = 5$?

**(A)** 3 **(B)** $2\sqrt{3}$ **(C)** 4 **(D)** $3\sqrt{3}$ **(E)** 6

**Answer (C):** By the converse of the Pythgorean Theorem, each face containing apex $A$ is a right triangle with right angle at $A$. Hence segments $\overline{AB}$, $\overline{AC}$, and $\overline{AD}$ are mutually perpendicular. The area of the $3-4-5$ right triangle $ACD$ is 6. If $\triangle ACD$ is considered to be the base of the tetrahedron, the height is $AB = 2$. The volume of the tetrahedron is therefore $\frac{1}{3} \cdot 6 \cdot 2 = 4$.




**Note:** There is a complicated formula, analogous to Heron's formula for the area of a triangle, giving the volume of a tetrahedron in terms of the edge lengths.

14. All the roots of polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of $B$?

**(A)** $-88$ **(B)** $-80$ **(C)** $-64$ **(D)** $-41$ **(E)** $-40$

**Answer (A):** Because this polynomial has degree 6, there are 6 roots, counting multiplicities. By Vieta's formulas, the sum of the roots is 10 and their product is 16. The only way this can happen is for the roots, listed with repetitions, to be $1, 1, 2, 2, 2, 2$. Thus the polynomial is $(z-1)^2(z-2)^4$. By the Binomial Theorem, this polynomial equals

$$(z^2 - 2z + 1) \cdot (z^4 - 8z^3 + 24z^2 - 32z + 16).$$

When this product is expanded, the coefficient of $z^3$ is $B = -32 - 48 - 8 = -88$.

**OR**

Proceed as in the first solution. Then, using Vieta's formulas, observe that $-B$ is the sum of the products of the roots taken 3 at a time, namely

$$\binom{4}{3}(2 \cdot 2 \cdot 2) + \binom{4}{2}\binom{2}{1}(2 \cdot 2 \cdot 1) + \binom{4}{1}\binom{2}{2}(2 \cdot 1 \cdot 1).$$

This gives a total of $32 + 48 + 8 = 88$. Therefore $B = -88$.

15. Values for $A$, $B$, $C$, and $D$ are to be selected from $\{1, 2, 3, 4, 5, 6\}$ without replacement (i.e., no two letters have the same value). How many ways are there to make such choices so that the two curves $y = Ax^2 + B$ and $y = Cx^2 + D$ intersect? (The order in which the curves are listed does not matter; for example, the choices $A = 3$, $B = 2$, $C = 4$, $D = 1$ is considered the same as the choices $A = 4$, $B = 1$, $C = 3$, $D = 2$.)

**(A)** 30 **(B)** 60 **(C)** 90 **(D)** 180 **(E)** 360

**Answer (C):** All these parabolas open upward and are symmetric about the $y$-axis. Because the selection of the coefficients for the two parabolas is made without replacement, the vertex and the narrowness of the first parabola are different from the vertex and the narrowness of the second parabola. The two parabolas intersect if and only if the vertex of the narrower parabola lies below the vertex of the wider parabola—the narrower one has the greater $x^2$ coefficient and the higher one is the one with the greater constant term. Therefore an intersection will occur if and only if $A - C$ and $B - D$ have opposite signs. There are $\binom{6}{2} = 15$ choices for an unordered set of 2 numbers from $\{1, 2, 3, 4, 5, 6\}$. Without loss of generality, let $A$ be the greater and $C$ the lesser, so that $A - C > 0$. There are $\binom{4}{2} = 6$ choices for an unordered set of 2 numbers from the 4 remaining numbers. The parabolas will intersect if and only if $B$ is the lesser of the two, so that $B - D < 0$. Thus there are $15 \cdot 6 = 90$ choices in all.

16. In the following list of numbers, the integer $n$ appears $n$ times in the list for $1 \le n \le 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, 200, \ldots, 200$$

What is the median of the numbers in this list?

**(A)** 100.5 **(B)** 134 **(C)** 142 **(D)** 150.5 **(E)** 167

**Answer (C):** There are

$$1 + 2 + 3 + \cdots + 200 = \frac{200 \cdot 201}{2} = 20{,}100$$

numbers in the list, so the median is the average (mean) of the two middle numbers, the 10,050th and 10,051st entries. The number of entries less than or equal to $n$ is

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Setting

$$\frac{n(n+1)}{2} = 10{,}050$$

and solving for $n$ suggests that $n$ is a little less than

$$\sqrt{20{,}100} \approx 100 \cdot 2.01^{\frac{1}{2}} \approx 100\left(\sqrt{2} + \frac{1}{2} \cdot 0.01\right) \approx 100(1.414 + 0.005) \approx 142.$$

A little arithmetic shows that

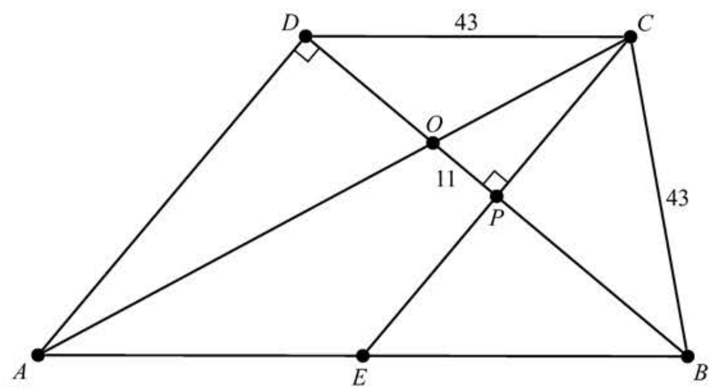
$$\frac{141(141+1)}{2} = 10{,}011 < 10{,}050 \quad \text{and} \quad \frac{142(142+1)}{2} = 10{,}153 > 10{,}051,$$

so both of the middle numbers in the list are 142, and this is the median.

17. Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}$, $BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let $O$ be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let $P$ be the midpoint of $\overline{BD}$. Given that $OP = 11$, the length $AD$ can be written in the form $m\sqrt{n}$, where $m$ and $n$ are positive integers and $n$ is not divisible by the square of any prime. What is $m + n$ ?

**(A)** 65 **(B)** 132 **(C)** 157 **(D)** 194 **(E)** 215

**Answer (D):** Note that $\triangle BCD$ is isosceles with vertex angle $C$, and $\overline{CP}$ is a median. Thus $\overline{CP} \perp \overline{BD}$. Also, because $\overline{AD} \perp \overline{BD}$, it follows that $\overline{CP} \parallel \overline{AD}$.




Let $E$ be the intersection of the lines $CP$ and $AB$. Then $AECD$ is a parallelogram, so $AE = CD = 43$ and $\angle DAE \cong \angle DCE$. But $\angle DCE \cong \angle ECB$, because $\overline{CP}$ is also an angle bisector in $\triangle BCD$, and $\angle DAE \cong \angle CEB$, as corresponding angles, because $\overline{CE} \parallel \overline{AD}$. By transitivity, $\angle ECB \cong \angle CEB$, so $\triangle BCE$ is isosceles with $EB = CB = 43$. Thus $AB = AE + EB = 43 + 43 = 86$.

Now $\triangle COD \sim \triangle AOB$, so $\frac{DO}{BO} = \frac{CD}{AB} = \frac{1}{2}$. Hence $DO = \frac{1}{3}BD$. This implies that

$$11 = OP = DP - DO = \frac{1}{2}BD - \frac{1}{3}BD = \frac{1}{6}BD,$$

so $BD = 66$.

The Pythagorean Theorem in $\triangle ABD$ gives

$$AD = \sqrt{AB^2 - BD^2} = \sqrt{86^2 - 66^2} = \sqrt{20 \cdot 152} = \sqrt{4 \cdot 5 \cdot 8 \cdot 19} = 4\sqrt{190}.$$

Thus $m = 4$, $n = 190$, and $m + n = 194$.

**OR**

Note that $\triangle AOD$ and $\triangle COP$ are similar, so $AD : CP = DO : 11$. Also, $\triangle CDB$ is isosceles, so $\angle PBC = \angle CDB = \angle DBA$. Therefore $\triangle DBA$ and $\triangle PBC$ are also similar. Thus $AD : CP = DB : PB = 2 : 1$. It follows that $DO = 22$, $DP = 33$, and $BD = 66$. Because $\triangle CDO$ and $\triangle ABO$ are similar with $DO : OB = 1 : 2$ it follows that $AB = 86$. The solution concludes as above.

18. Let $f$ be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers $a$ and $b$. Suppose that $f$ also has the property that $f(p) = p$ for every prime number $p$. For which of the following numbers $x$ is $f(x) < 0$?

**(A)** $\frac{17}{32}$ **(B)** $\frac{11}{16}$ **(C)** $\frac{7}{9}$ **(D)** $\frac{7}{6}$ **(E)** $\frac{25}{11}$

**Answer (E):** If $n$ is a positive integer whose prime factorization is $n = p_1 p_2 \cdots p_k$, then $f(n) = f(p_1) + f(p_2) + \cdots + f(p_k) = p_1 + p_2 + \cdots + p_k$. Because $f(1) = f(1 \cdot 1) = f(1) + f(1)$, it follows that $f(1) = 0$. If $r$ is a positive rational number, then $0 = f(1) = f\left(r \cdot \frac{1}{r}\right) = f(r) + f\left(\frac{1}{r}\right)$, which implies that $f\left(\frac{1}{r}\right) = -f(r)$ for all positive rational numbers $r$. Hence

$$f\left(\frac{17}{32}\right) = 17 - 5 \cdot 2 = 7 > 0,$$
$$f\left(\frac{11}{16}\right) = 11 - 4 \cdot 2 = 3 > 0,$$
$$f\left(\frac{7}{9}\right) = 7 - 2 \cdot 3 = 1 > 0,$$
$$f\left(\frac{7}{6}\right) = 7 - 2 - 3 = 2 > 0, \text{ and}$$
$$f\left(\frac{25}{11}\right) = 2 \cdot 5 - 11 = -1 < 0.$$

Of the choices, only $x = \frac{25}{11}$ has the property that $f(x) < 0$.

19. The area of the region bounded by the graph of

$$x^2 + y^2 = 3|x - y| + 3|x + y|$$

is $m + n\pi$, where $m$ and $n$ are integers. What is $m + n$?

**(A)** 18 **(B)** 27 **(C)** 36 **(D)** 45 **(E)** 54

**Answer (E):** The lines $y = x$ and $y = -x$ divide the coordinate plane into four regions. In the region including the positive $x$-axis, $x - y > 0$ and $x + y > 0$, so

$$x^2 + y^2 = 3(x - y) + 3(x + y),$$

which is equivalent to

$$(x - 3)^2 + y^2 = 9.$$

In the region including the positive $y$-axis, $x - y < 0$ and $x + y > 0$, so

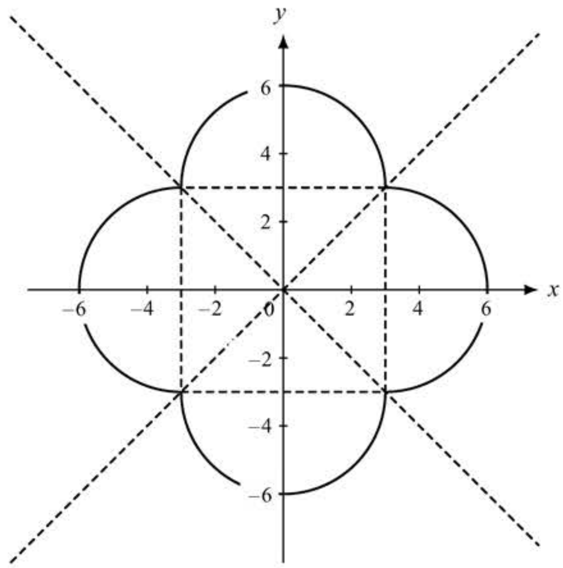
$$x^2 + y^2 = -3(x - y) + 3(x + y),$$

which is equivalent to

$$x^2 + (y - 3)^2 = 9.$$

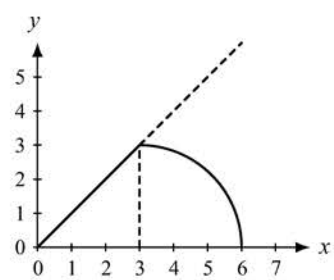
Similarly, in the region including the negative $x$-axis, the defining equation is equivalent to $(x + 3)^2 + y^2 = 9$, and in the region including the negative $y$-axis, the defining equation is equivalent to $x^2 + (y + 3)^2 = 9$.

The complete graph, shown below, therefore consists of four semicircles whose diameters form a square of side length 6. The enclosed area is thus $36 + 4 \cdot \frac{1}{2}\pi \cdot 3^2 = 36 + 18\pi$. The requested sum is $36 + 18 = 54$.




**OR**

The given curve is symmetric with respect to the $x$-axis, with respect to the $y$-axis, with respect to the line $y = x$, and with respect to the line $y = -x$. Hence the area in question is 8 times the area of that part of the region that lies in the wedge $0 < y < x$, bounded by the curve $x^2 + y^2 = 3x - 3y + 3x + 3y = 6x$. This is equivalent to $(x-3)^2 + y^2 = 3^2$ and is a circle with center $(3, 0)$ and radius 3. As seen in the figure below, this gives rise to a triangle with vertices at $(0, 0)$, $(3, 0)$, and $(3, 3)$ together with a quarter circle of radius 3 centered at $(3, 0)$ extending from $(6, 0)$ counterclockwise to $(3, 3)$ with vertices $(3, 3)$, $(3, 0)$, and $(6, 0)$. The area of this region is $\frac{9}{2} + \frac{9}{4}\pi$. The total area is 8 times this, which is $36 + 18\pi$, and the requested sum is $36 + 18 = 54$.




20. In how many ways can the sequence $1, 2, 3, 4, 5$ be rearranged so that no three consecutive terms are increasing and no three consecutive terms are decreasing?

**(A)** 10 **(B)** 18 **(C)** 24 **(D)** 32 **(E)** 44

**Answer (D):** Let $A$ be the set of sequences in which the first three terms are increasing; let $B$ be the set of sequences in which the second, third, and fourth terms are increasing; and let $C$ be the set of sequences in which the last three terms are increasing. Then $|A| = |B| = |C| = \binom{5}{3}\cdot 2! = 20$. Also, $|A \cap B| = |B \cap C| = \binom{5}{4}\cdot 1 = 5$. Finally, $|A \cap C| = |A \cap B \cap C| = \binom{5}{5} = 1$. By the Inclusion-Exclusion Principle, the number of sequences that have at least 3 consecutive increasing terms is $20 + 20 + 20 - 5 - 5 - 1 + 1 = 50$.

By symmetry, there are 50 sequences with at least three consecutive decreasing terms. For a sequence to have both three consecutive increasing and three consecutive decreasing terms (and thus be counted in both sets of 50), one of two possibilities occurs.

- The first three terms must increase and the last three terms must decrease, with the largest number, 5, as the third term. There are $\binom{4}{2}$ possible choices for the first two numbers, which determines the sequence in its entirety, so there are 6 such sequences.
- The first three terms must decrease and the last three terms must increase, with the smallest number, 1, as the third term. By analogous reasoning to the previous case, there are 6 such sequences.

Therefore the number of sequences that have either three consecutive increasing terms or three consecutive decreasing terms is $50 + 50 - 12 = 88$. There are $5! = 120$ sequences in all, so there are $120 - 88 = 32$ sequences that have neither three increasing consecutive terms nor three consecutive decreasing terms.

**OR**

Consider the various orders in which 1, 2, and 3 can appear:

- If they appear in the order 123, then the only ways to insert 4 and 5 so as to satisfy the conditions of the problem are 14253, 15243, 41523, and 51423, for 4 ways.
- If they appear in the order 132, then there are 6 ways to insert 4 and 5: 41325, 51324, 45132, 13254, 14352, and 15342.
- If they appear in the order 213, there are again 6 ways to insert 4 and 5: 42513, 52413, 24153, 25143, 21435, and 21534.

Each of the other orders for 1, 2, and 3 is a reversal of one of the above, so the total number of ways is $2(4 + 6 + 6) = 32$.

**OR**

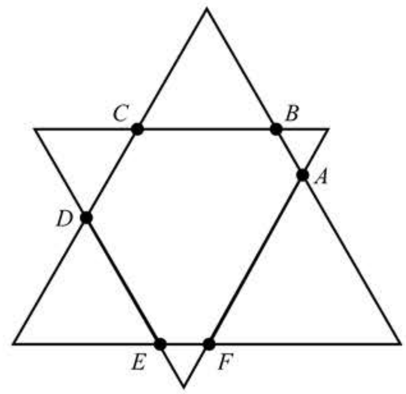
To satisfy the conditions of the problem, the sequence $a_1, a_2, a_3, a_4, a_5$ must satisfy either $a_1 < a_2 > a_3 < a_4 > a_5$ or $a_1 > a_2 < a_3 > a_4 < a_5$. If in an ordering of the first sort, each number, say $k$, is replaced by $6 - k$, then all inequalities are reversed, which produces an ordering of the second type, and vice versa, so the two types of ordering occur the same number of times.

Consider the first type of ordering. There are two local maxima and three local minima. The numbers 1 and 2 cannot be local maxima, so the local maxima are two of 3, 4, and 5. Moreover, the number 5 must be one of the local maxima, because 5 cannot be a local minimum. Thus the two maxima are either 3 and 5, or 4 and 5. By considering left-right symmetry, it may be assumed that the smaller one is to the left, and the larger one to the right; this gives a factor 2 in the final count. So the sequence can be $*3*5*$ or $*4*5*$. In the first case there is only one possible location for 4, namely last, and the first and third positions are the numbers 1 and 2 in either order, which gives 2 possibilities. In the second case, the asterisks are the numbers 1, 2, and 3 in any order, which gives 6 possibilities. In all there are $2 \cdot 2 \cdot (2 + 6) = 32$ such sequences.

21. Let $ABCDEF$ be an equiangular hexagon. The lines $AB$, $CD$, and $EF$ determine a triangle with area $192\sqrt{3}$, and the lines $BC$, $DE$, and $FA$ determine a triangle with area $324\sqrt{3}$. The perimeter of hexagon $ABCDEF$ can be expressed as $m + n\sqrt{p}$, where $m$, $n$, and $p$ are positive integers and $p$ is not divisible by the square of any prime. What is $m + n + p$?

**(A)** 47 **(B)** 52 **(C)** 55 **(D)** 58 **(E)** 63

**Answer (C):** Because the given hexagon is equiangular, each of its interior angles measures 120°, so each angle adjacent to one of these measures 60°. This forces the three angles in every triangle in the figure shown below to be 60°.




Let $x = BC + DE + FA$ and $y = AB + CD + EF$. Then the perimeter of the hexagon is $x + y$. The triangle determined by lines $AB$, $CD$, and $EF$ has perimeter $2x + y$, and the triangle determined by lines $BC$, $DE$, and $FA$ has perimeter $x + 2y$. Then

$$\frac{\sqrt{3}}{4}\left(\frac{2x+y}{3}\right)^2 = 192\sqrt{3} \quad \text{and} \quad \frac{\sqrt{3}}{4}\left(\frac{x+2y}{3}\right)^2 = 324\sqrt{3}.$$

Thus

$$\frac{2x+y}{3} = \sqrt{768} = 16\sqrt{3} \quad \text{and} \quad \frac{x+2y}{3} = \sqrt{1296} = 36.$$

Adding these equations gives $x + y = 36 + 16\sqrt{3}$. The requested sum is $36 + 16 + 3 = 55$.

22. Hiram's algebra notes are 50 pages long and are printed on 25 sheets of paper; the first sheet contains pages 1 and 2, the second sheet contains pages 3 and 4, and so on. One day he leaves his notes on the table before leaving for lunch, and his roommate decides to borrow some pages from the middle of the notes. When Hiram comes back, he discovers that his roommate has taken a consecutive set of sheets from the notes and that the average (mean) of the page numbers on all remaining sheets is exactly 19. How many sheets were borrowed?

**(A)** 10 **(B)** 13 **(C)** 15 **(D)** 17 **(E)** 20

**Answer (B):** Let $a$ and $b$ be even positive integers with $1 \le a \le b \le 50$ such that

$$1, 2, \ldots, a, b+1, b+2, \ldots, 50$$

are the numbers of the pages that remain. The sum of these numbers is

$$\begin{aligned}(1 + 2 + &\cdots + a) + ((b+1) + \cdots + 50)\\ &= (1 + 2 + \cdots + 50) - ((a+1) + \cdots + b)\\ &= \frac{50 \cdot 51}{2} - (1 + 2 + \cdots + b) + (1 + 2 + \cdots + a)\\ &= \frac{50 \cdot 51}{2} - \frac{b(b+1)}{2} + \frac{a(a+1)}{2}.\end{aligned}$$

The number of pages that remain is $50 - b + a$. Therefore

$$19 = \frac{\frac{50\cdot 51}{2} - \frac{b(b+1)}{2} + \frac{a(a+1)}{2}}{50 - b + a} = \frac{50 \cdot 51 - b(b+1) + a(a+1)}{2(50 - b + a)}.$$

Simplifying yields

$$50 \cdot 13 = (b^2 - a^2) - 37(b - a) = (b - a)(b + a - 37).$$

Observe that $50 \cdot 13 = 2 \cdot 5^2 \cdot 13$, and that $b - a$ and $b + a - 37$ are two positive integers with sum equal to $2b - 37 \le 100 - 37 = 63$. The only pairs of factors that sum to at most 63 are $\{13, 50\}$ and $\{26, 25\}$. Testing these gives the four possible solutions for $(a, b)$, namely $(0, 50)$, $(18, 44)$, $(19, 44)$, and $(37, 50)$. The only solution in which both $a$ and $b$ are positive even integers is $(18, 44)$. Thus Hiram's roommate borrowed the sheets containing pairs of page numbers $\{19, 20\}, \{21, 22\}, \ldots, \{43, 44\}$. There are $\frac{44-18}{2} = 13$ sheets in all.

23. Frieda the frog begins a sequence of hops on a $3 \times 3$ grid of squares, moving one square on each hop and choosing at random the direction of each hop—up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example if Frieda begins in the center square and makes two hops "up", the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

**(A)** $\frac{9}{16}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{25}{32}$ **(E)** $\frac{13}{16}$

**Answer (D):** Let $m$ denote the middle square, $c$ denote a corner square, and $e$ denote an edge square not in the corner. There are four ways to reach a corner in at most four moves starting from $m$:

$$ec\text{, with probability } 1 \cdot \tfrac{1}{2} \cdot = \tfrac{1}{2},$$
$$emec\text{, with probability } 1 \cdot \tfrac{1}{4} \cdot 1 \cdot \tfrac{1}{2} = \tfrac{1}{8},$$
$$eec\text{, with probability } 1 \cdot \tfrac{1}{4} \cdot \tfrac{1}{2} = \tfrac{1}{8}\text{, and}$$
$$eeec\text{, with probability } 1 \cdot \tfrac{1}{4} \cdot \tfrac{1}{4} \cdot \tfrac{1}{2} = \tfrac{1}{32},$$

for a total probability of $\frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{32} = \frac{25}{32}$.

**OR**

The matrices below show the number of ways to reach each square after 1, 2, 3, and 4 hops.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 5 & 2 \\ 5 & 4 & 5 \\ 2 & 5 & 2 \end{bmatrix} \quad \begin{bmatrix} 10 & 9 & 10 \\ 9 & 20 & 9 \\ 10 & 9 & 10 \end{bmatrix}$$

The first matrix shows that after one hop Frieda will be on one of the side edge squares. There is exactly one way to reach each of those squares in one hop.

The second matrix shows the number of ways to reach each square in two hops. For example to reach the upper left corner, Frieda can come from the right or from below so the two 1s adjacent to the corner in the first matrix are added to make 2 in the second matrix. It follows that Frieda can reach one of the corner squares in two hops in $4 \cdot 2 = 8$ different ways out of $4^2 = 16$ possible two-hop sequences.

Now consider the third hop assuming that a corner square has not yet been reached, making sure to count the wrap-around hops. For example the top row middle square can be reached from the center square or from the bottom row middle square (but not from the adjacent corner squares). The values 4 and 1 are added from the second matrix to make 5 in the third matrix. The matrix shows that Frieda can reach a corner square on her third hop in $4 \cdot 2 = 8$ different ways out of $4^3 = 64$ possible three-hop sequences.

Finally the fourth matrix shows that Frieda can reach a corner square on her fourth hop in $4 \cdot 10 = 40$ different ways out of $4^4 = 256$ possible four-hop sequences. The probability of landing on a corner square on one of the four hops is therefore

$$\frac{8}{16} + \frac{8}{64} + \frac{40}{256} = \frac{25}{32}.$$

24. The interior of a quadrilateral is bounded by the graphs of $(x + ay)^2 = 4a^2$ and $(ax - y)^2 = a^2$, where $a$ is a positive real number. What is the area of this region in terms of $a$, valid for all $a > 0$?

**(A)** $\frac{8a^2}{(a+1)^2}$ **(B)** $\frac{4a}{a+1}$ **(C)** $\frac{8a}{a+1}$ **(D)** $\frac{8a^2}{a^2+1}$ **(E)** $\frac{8a}{a^2+1}$

**Answer (D):** Taking the positive and negative square roots of both sides of these equations shows that this region is bounded by the graphs of four lines: $x + ay = \pm 2a$ and $ax - y = \pm a$. Because their slopes are negative reciprocals of each other, lines $x + ay = 2a$ and $ax - y = a$ are perpendicular, as are lines $x + ay = -2a$ and $ax - y = -a$. Therefore the region bounded by these lines is the interior of a rectangle.

The distance between two parallel lines $Ax + By = C_1$ and $Ax + By = C_2$ is

$$\frac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}.$$

Thus the distance between the first two lines is

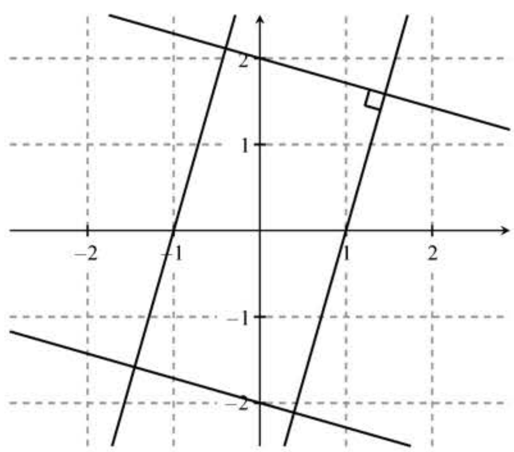
$$\frac{2a - (-2a)}{\sqrt{1^2 + a^2}} = \frac{4a}{\sqrt{a^2 + 1}}.$$

The distance between the last two lines is

$$\frac{a - (-a)}{\sqrt{1^2 + a^2}} = \frac{2a}{\sqrt{a^2 + 1}}.$$

The area of the rectangle is the product of these two distances, $\frac{8a^2}{a^2+1}$.

**Note:** The shape of the rectangle depends on $a$. The graph below shows the situation when $a = 3.5$, with area of approximately 7.4.




25. How many ways are there to place 3 indistinguishable red chips, 3 indistinguishable blue chips, and 3 indistinguishable green chips in the squares of a $3 \times 3$ grid so that no two chips of the same color are directly adjacent to each other, either vertically or horizontally?

**(A)** 12 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 36

**Answer (E):** The crucial claim is that the configuration must match one of the two configurations shown below up to a permutation of the colors of the chips and up to a rotation of the board. This implies that the answer is $(4 + 2) \cdot 3! = 36$.

| | | |
|---|---|---|
| R | G | R |
| B | R | B |
| G | B | G |

| | | |
|---|---|---|
| R | B | G |
| G | R | B |
| B | G | R |

Without loss of generality, let the chip in the center square be red. By the adjacency condition, the other two red chips must appear in the corner squares of the grid. There are two possibilities for the positions of these chips: either they are adjacent to the same side or they lie along a main diagonal.

In the first case, without loss of generality, assume that the two remaining red chips are on the top row and that a blue chip is placed in the bottom center square. Then the adjacency condition forces the two remaining squares on the bottom row to be filled with green chips, and the rest of the board is forced.

In the second case, without loss of generality, assume that the two remaining red chips are in the upper left and lower right squares and that a green chip is in the upper right square. Again the rest of the board is forced.

**OR**

There are two cases to consider: no color is repeated in the middle row or there are two chips of the same color in the middle row. Without loss of generality, if the middle row is GRB (see the figure on the right, above), then there are four possibilities for the middle column: BRB, BRG, GRB, and GRG. Without loss of generality, if the middle row is BRB (see the figure on the left, above), then there are two possibilities for the middle column: BRG and GRB. That is a total of 6 configurations, and permuting the colors then gives a total of $3! \cdot 6 = 36$ configurations.

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