MAA AMC

American Mathematics Competitions

Official Solutions

MAA American Mathematics Competitions

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AMC 10 A

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FALL

This official solutions booklet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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The problems and solutions for this AMC 10 A were prepared by the
MAA AMC 10/12 Editorial Board under the direction of
Azar Khosravani and Carl Yerger, co-Editors-in-Chief.

1. What is the value of $\frac{(2112-2021)^2}{169}$?

   (A) 7 **(B)** 21 **(C)** 49 **(D)** 64 **(E)** 91

   Answer (C): Observe that $2112 - 2021 = 91 = 7 \cdot 13$ and that $169 = 13^2$. Thus
$$\frac{(2112-2021)^2}{169} = \frac{(7 \cdot 13)^2}{13^2} = 7^2 = 49.$$

2. Menkara has a 4×6 index card. If she shortens the length of one side of this card by 1 inch, the card would have area 18 square inches. What would the area of the card be in square inches if instead she shortens the length of the other side by 1 inch?

   (A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

   Answer (E): Shortening the side with length 4 by 1 inch yields a 3×6 card, whose area is 18 square inches. Shortening the side with length 6 by 1 inch results in a 4×5 card, whose area is 20 square inches.

3. What is the maximum number of balls of clay with radius 2 that can completely fit inside a cube of side length 6 assuming that the balls can be reshaped but not compressed before they are packed in the cube?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

   Answer (D): The volume of each ball of radius 2 is $\frac{4}{3} \cdot \pi \cdot 2^3 = \frac{32}{3}\pi$. The volume of the cube of side length 6 is $6^3 = 216$. There is room for at most
$$\frac{216}{\frac{32}{3}\pi} = \frac{648}{32\pi} = \frac{81}{4\pi}$$
   balls. Because $12 < 4\pi < 13$, it follows that
$$6 < \frac{81}{13} < \frac{81}{4\pi} < \frac{81}{12} < 7.$$
   Hence the maximum number of reshaped balls that can completely fit inside the cube is 6.

4. Mr. Lopez has a choice of two routes to get to work. Route A is 6 miles long, and his average speed along this route is 30 miles per hour. Route B is 5 miles long, and his average speed along this route is 40 miles per hour, except for a $\frac{1}{2}$-mile stretch in a school zone where his average speed is 20 miles per hour. By how many minutes is Route B quicker than Route A?

   (A) $2\frac{3}{4}$ **(B)** $3\frac{3}{4}$ **(C)** $4\frac{1}{2}$ **(D)** $5\frac{1}{2}$ **(E)** $6\frac{3}{4}$

   Answer (B): The time required for Route A is 6 miles divided by 30 miles per hour, which is $\frac{1}{5}$ hour. The time required for Route B is $5 - \frac{1}{2} = \frac{9}{2}$ miles divided by 40 miles per hour, plus $\frac{1}{2}$ mile divided by 20 miles per hour, which is $\frac{9}{80} + \frac{1}{40} = \frac{11}{80}$ hour. The difference in time is
$$\frac{1}{5} - \frac{11}{80} = \frac{16-11}{80} = \frac{5}{80} = \frac{1}{16}$$
   hour. This is equivalent to $\frac{1}{16} \cdot 60 = \frac{15}{4} = 3\frac{3}{4}$ minutes.

5. The six-digit number $\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}\,\underline{0}\,\underline{A}$ is prime for only one digit A. What is A ?

   (A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

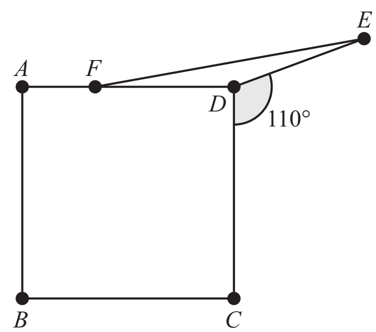
   Answer (E): A number whose units digit is 5 is divisible by 5, so choice **(C)** does not give a prime. Choices **(A)** and **(D)** can be ruled out by the fact that a number is divisible by 3 if and only if the sum of its digits is divisible by 3; here $2+0+2+1+0+1=6$ and $2+0+2+1+0+7=12$ are divisible by 3. Choice **(B)** can be ruled out by the fact that a number is divisible by 11 if and only if the alternating sum of its digits is divisible by 11; here $2-0+2-1+0-3=0$ is divisible by 11. The five even choices for A result in a number divisible by 2. Given that one choice produces a prime, A must equal 9 and the number 202109 must be prime. (In fact, this can be verified with computer algebra software.)

6. Elmer the emu takes 44 equal strides to walk between consecutive telephone poles on a rural road. Oscar the ostrich can cover the same distance in 12 equal leaps. The telephone poles are evenly spaced, and the 41st pole along this road is exactly one mile (5280 feet) from the first pole. How much longer, in feet, is Oscar's leap than Elmer's stride?

   (A) 6 **(B)** 8 **(C)** 10 **(D)** 11 **(E)** 15

   Answer (B): Consecutive telephone poles are $\frac{5280}{40}=132$ feet apart. Elmer's stride is then $\frac{132}{44}=3$ feet long, and Oscar's leap is $\frac{132}{12}=11$ feet long. The requested difference is $11-3=8$.

7. As shown in the figure below, point E lies in the opposite half-plane determined by line CD from point A so that $\angle CDE=110°$. Point F lies on $\overline{AD}$ so that $DE=DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



   (A) 160 **(B)** 164 **(C)** 166 **(D)** 170 **(E)** 174

   Answer (D): Note that $\angle EDF=360°-\angle ADC-\angle CDE=360°-90°-110°=160°$. Because $\triangle DEF$ is isosceles, angles DEF and DFE have an equal measure of $\frac{180°-160°}{2}=10°$. Hence $\angle AFE=180°-\angle DFE=180°-10°=170°$.

8. A two-digit positive integer is said to be *cuddly* if it is equal to the sum of its nonzero tens digit and the square of its units digit. How many two-digit positive integers are cuddly?

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Answer (B): The value of a two-digit cuddly number $\underline{a}\,\underline{b}$ is $10a+b$. Therefore $10a+b=a+b^2$, so $9a=b(b-1)$. It follows that 9 divides $b(b-1)$. If $3\mid b$ and $3\mid(b-1)$, then $3\mid 1$, which is a contradiction. Thus either $9\mid b$ or $9\mid(b-1)$. Furthermore, $a\neq 0$ implies $b>1$. Therefore the only possible value for b is 9. Then $9a=9\cdot 8$, so $a=8$, and the only two-digit cuddly number is 89.

9. When a certain unfair die is rolled, an even number is 3 times as likely to appear as an odd number. The die is rolled twice. What is the probability that the sum of the numbers rolled is even?

   (A) $\frac{3}{8}$ **(B)** $\frac{4}{9}$ **(C)** $\frac{5}{9}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{5}{8}$

   Answer (E): Suppose that the probability of rolling an odd number is p. The probability of rolling an even number is then $3p$. Because $p+3p=1$, it follows that $p=\frac{1}{4}$, so the probability of rolling an odd number is $\frac{1}{4}$ and the probability of rolling an even number is $1-\frac{1}{4}=\frac{3}{4}$. The sum will be even if both rolls are even or both rolls are odd. Therefore the probability of rolling an even sum is $\frac{3}{4}\cdot\frac{3}{4}+\frac{1}{4}\cdot\frac{1}{4}=\frac{5}{8}$.

10. A school has 100 students and 5 teachers. In the first period, each student is taking one class, and each teacher is teaching one class. The enrollments in the classes are 50, 20, 20, 5, and 5. Let t be the average value obtained if a teacher is picked at random and the number of students in their class is noted. Let s be the average value obtained if a student is picked at random and the number of students in their class, including that student, is noted. What is $t - s$?

(A) -18.5 **(B)** -13.5 **(C)** 0 **(D)** 13.5 **(E)** 18.5

Answer (B): The average class size when a teacher—that is, a class—is picked at random is given by

$$t = \frac{1}{5} \cdot 50 + \frac{2}{5} \cdot 20 + \frac{2}{5} \cdot 5 = 20.$$

A randomly picked student has a $\frac{50}{100}$ chance of being in the class with 50 students, a $\frac{40}{100}$ chance of being in a class with 20 students, and a $\frac{10}{100}$ chance of being in a class with 5 students, so

$$s = \frac{50}{100} \cdot 50 + \frac{40}{100} \cdot 20 + \frac{10}{100} \cdot 5 = 33.5.$$

The requested difference is $20 - 33.5 = -13.5$.

11. Emily sees a ship traveling at a constant speed along a straight section of a river. She walks parallel to the riverbank at a uniform rate faster than the ship. She counts 210 equal steps walking from the back of the ship to the front. Walking in the opposite direction, she counts 42 steps of the same size from the front of the ship to the back. In terms of Emily's equal steps, what is the length of the ship?

(A) 70 **(B)** 84 **(C)** 98 **(D)** 105 **(E)** 126

Answer (A): Let L denote the required length of the ship. Assume Emily's speed is 1 and let $v < 1$ denote the speed of the ship relative to the riverbank. When Emily walks from the back to the front, it takes 210 steps, so $210(1 - v) = L$. Similarly, when Emily walks from the front to the back of the ship, it takes 42 steps, so $42(1 + v) = L$. Solving this system yields $v = \frac{2}{3}$ and $L = 70$.

12. The base-nine representation of the number N is $27{,}006{,}000{,}052_{\text{nine}}$. What is the remainder when N is divided by 5 ?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Answer (D): Note that $N = 2 \cdot 9^{10} + 7 \cdot 9^9 + 6 \cdot 9^6 + 5 \cdot 9^1 + 2 \cdot 9^0$. Because even powers of 9 leave remainder 1 when divided by 5 and odd powers of 9 leave remainder congruent to -1 when divided by 5, it follows that N leaves remainder congruent to $2 - 7 + 6 - 5 + 2 = -2 \equiv 3 \pmod 5$, so the requested remainder is 3.

13. Each of 6 balls is randomly and independently painted either black or white with equal probability. What is the probability that every ball is different in color from more than half of the other 5 balls?

(A) $\frac{1}{64}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{5}{16}$ **(E)** $\frac{1}{2}$

Answer (D): The specified event will occur if and only if there are 3 balls of each color. Indeed, if they are painted that way, then each ball has a different color than $\frac{3}{5}$ of the other balls. Conversely, if 4 or more balls are black, or 4 or more are white, then each of those balls has the same color as at least $\frac{3}{5}$ of the other balls.

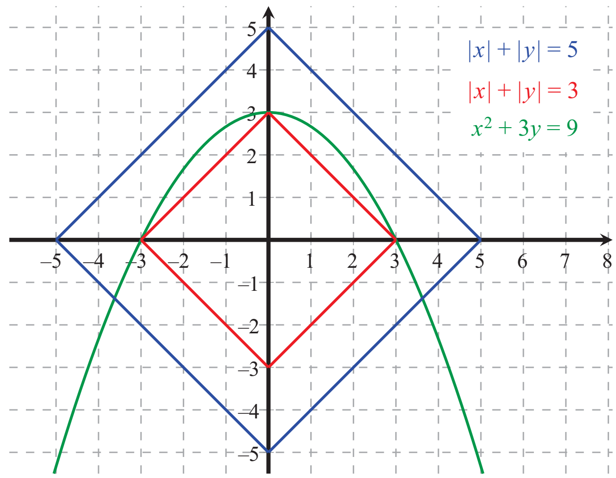
The number of ways to paint the balls so that there are 3 balls of each color is the number of ways to choose 3 of the 6 balls to be white, which is $\binom{6}{3} = 20$. There are $2^6 = 64$ equally likely ways to paint the balls, so the requested probability is equal to $\frac{20}{64} = \frac{5}{16}$.

14. How many ordered pairs (x, y) of real numbers satisfy the following system of equations?

$$x^2 + 3y = 9$$
$$(|x| + |y| - 4)^2 = 1$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 7

Answer (D): The graph of the first equation is a parabola opening downward with vertex $(0, 3)$, passing through $(-3, 0)$ and $(3, 0)$. The second equation is satisfied if either $|x| + |y| = 5$ or $|x| + |y| = 3$. Therefore the graph of the second equation is a square with vertices $(5, 0)$, $(0, 5)$, $(-5, 0)$, and $(0, -5)$ together with a square with vertices $(3, 0)$, $(0, 3)$, $(-3, 0)$, and $(0, -3)$. As shown below, the two graphs intersect at 5 points—$(0, 3)$, $(3, 0)$, $(-3, 0)$, and 2 points on the larger square in the lower half-plane.



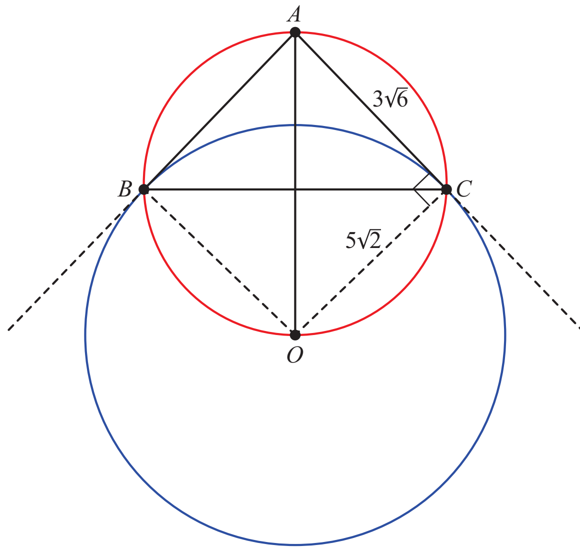
15. Isosceles triangle ABC has $AB = AC = 3\sqrt{6}$, and a circle with radius $5\sqrt{2}$ is tangent to line AB at B and to line AC at C. What is the area of the circle that passes through vertices A, B, and C ?

(A) 24π **(B)** 25π **(C)** 26π **(D)** 27π **(E)** 28π

Answer (C): Let O be the center of the circle with radius $5\sqrt{2}$. Consider the circle with diameter $\overline{AO}$. Because $\angle ABO$ and $\angle ACO$ are right angles, the opposite angles of quadrilateral $ABOC$ are supplementary, and hence this quadrilateral is cyclic. Thus O is also on the circle that passes through A, B, and C, and by symmetry $\overline{AO}$ is a diameter. By the Pythagorean Theorem,

$$AO = \sqrt{\left(5\sqrt{2}\right)^2 + \left(3\sqrt{6}\right)^2} = 2\sqrt{26},$$

so the circle that passes through A, B, and C has radius $\sqrt{26}$ and area 26π.



16. The graph of $f(x) = |\lfloor x \rfloor| - |\lfloor 1 - x \rfloor|$ is symmetric about which of the following? (Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

(A) the y-axis **(B)** the line $x = 1$ **(C)** the origin

(D) the point $\left(\frac{1}{2}, 0\right)$ **(E)** the point $(1, 0)$

Answer (D): Let $x = n + r$, where n is an integer and $0 \le r < 1$. First suppose that $r = 0$. Then $f(x) = |n| - |1 - n|$, which is equal to $n - (n - 1) = 1$ if $n \ge 1$ and $(-n) - (1 - n) = -1$ if $n \le 0$. Hence the graph of f cannot be symmetric about the y-axis, the line $x = 1$, the origin, or the point $(1, 0)$. However, note that in all cases $f(1 - n) = -f(n)$, so the graph of f is symmetric about the point $\left(\frac{1}{2}, 0\right)$ when x is restricted to integer values. Finally, when $r > 0$, observe that $\lfloor x \rfloor = n$ and $\lfloor 1 - x \rfloor = \lfloor 1 - n - r \rfloor = -n$, so $f(x) = |n| - |-n| = 0$, and again $f(1 - x) = -f(x)$. Hence the graph of f is symmetric about the point $\left(\frac{1}{2}, 0\right)$.

OR

Note that $f(1 - x) = \big|\lfloor 1 - x \rfloor\big| - \big|\lfloor 1 - (1 - x) \rfloor\big| = -f(x)$. Thus a point (x, y) is on the graph of f if and only if $(1 - x, -y)$ is on the graph. The midpoint of the segment joining these two points is the point $\left(\frac{1}{2}, 0\right)$; that is, the graph is symmetric about this point.

To show that $f(x)$ does not have any of the properties described in the four other alternatives, it is useful to note that

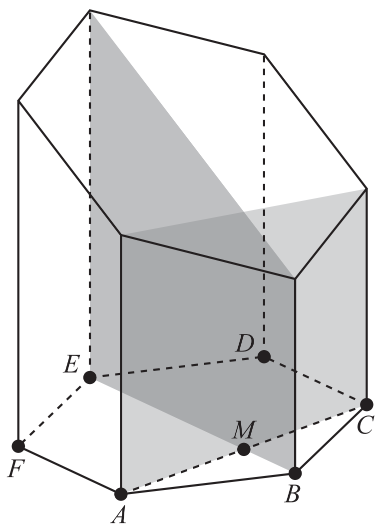
$$f(x) = \begin{cases} 1 & \text{if } n \text{ is an integer, } n \ge 1, \\ 0 & \text{if } n \text{ is not an integer,} \\ -1 & \text{if } n \text{ is an integer, } n \le 0. \end{cases}$$

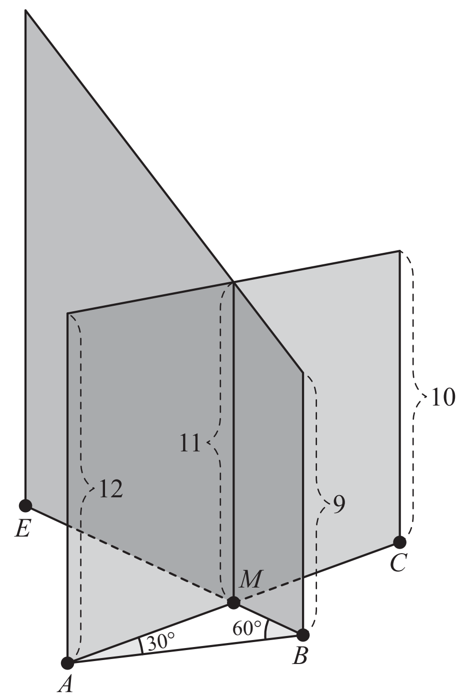
The graph of f is not symmetric about the y-axis because $f(1) = 1$ and $f(-1) = -1 \ne 1$. The graph of f is not symmetric about the line $x = 1$ because $f(3) = 1$ and $f(-1) = -1 \ne 1$. The graph of f is not symmetric with respect to the origin because $f(0) = 1$ and the reflected point $(0, -1)$ is not in the graph of f. The graph of f is not symmetric about the point $(1, 0)$ because $f(1) = 1$ and the reflected point $(1, -1)$ is not in the graph of f.

17. An architect is building a structure that will place vertical pillars at the vertices of regular hexagon $ABCDEF$, which is lying horizontally on the ground. The six pillars will hold up a flat solar panel that will not be parallel to the ground. The heights of the pillars at A, B, and C are 12, 9, and 10 meters, respectively. What is the height, in meters, of the pillar at E ?

(A) 9 **(B)** $6\sqrt{3}$ **(C)** $8\sqrt{3}$ **(D)** 17 **(E)** $12\sqrt{3}$

Answer (D): Let M be the midpoint of $\overline{AC}$. The pillars at A and C are 12 and 10 meters high, respectively, so the height of the panel at M is 11 meters. Because $\triangle BMA$ is a $30-60-90^\circ$ triangle, BM is half of the side length of the hexagon. Therefore $BM = \frac{1}{4}BE$. It follows that the height of the panel at point E is $9 + 4 \cdot (11 - 9) = 17$ meters.





18. A farmer's rectangular field is partitioned into a 2 by 2 grid of 4 rectangular sections as shown in the figure. In each section the farmer will plant one crop: corn, wheat, soybeans, or potatoes. The farmer does not want to grow corn and wheat in any two sections that share a border, and the farmer does not want to grow soybeans and potatoes in any two sections that share a border. Given these restrictions, in how many ways can the farmer choose crops to plant in each of the four sections of the field?

(A) 12 **(B)** 64 **(C)** 84 **(D)** 90 **(E)** 144

Answer (C): Let the sections of the field be numbered clockwise starting with the upper left as sections 1, 2, 3, and 4. There are 4 ways to select a crop to plant in section 1. If section 3 is planted with the crop in section 1, there are 3 ways to select crops for each of sections 2 and 4, yielding $4 \cdot 3 \cdot 3 = 36$ ways. If section 3 is planted with the crop that cannot be planted adjacent to the crop in section 1, then there are 2 ways to select crops for each of sections 2 and 4, yielding $4 \cdot 2 \cdot 2 = 16$ ways. If section 3 is planted with one of the other 2 crops, then there are 2 ways to select crops for each of sections 2 and 4, yielding $4 \cdot 2 \cdot 2 \cdot 2 = 32$ ways. This gives a total of $36 + 16 + 32 = 84$ ways.

OR

If only one crop is planted, then there are 4 ways to choose that crop. If two compatible crops are planted, then there are $2 \cdot 2 = 4$ ways to choose those crops and then $2^4 - 2 = 14$ ways to plant, for $4 \cdot 14 = 56$ possibilities. It is not possible to plant two incompatible crops without violating the condition. If three crops are planted, then there are 4 ways to choose them and $2 \cdot 2 = 4$ ways to plant, for 16 choices. If four crops are planted, then there are $4 \cdot 2 = 8$ ways to plant them. The total number of choices is $4 + 56 + 16 + 8 = 84$.

OR

As above, let the sections of the field be numbered clockwise starting with the upper left as sections 1, 2, 3, and 4. There are $4^4 = 256$ ways for crops to be assigned to sections of the field without regard to crops that are not to be planted in adjacent sections.

This problem will be solved via complementary counting. To this end, let A be the set of ways such that sections 1 and 2 are planted with crops that should not be planted in adjacent sections, and similarly, let B, C, and D be the set of ways sections 2 and 3, 3 and 4, and 4 and 1, respectively, can be planted with crops that should not be planted in adjacent sections. Each of the sets A, B, C, and D contains $2 \cdot 2 \cdot 4 \cdot 4 = 64$ elements. Each of the intersections $A \cap B$, $B \cap C$, $C \cap D$, and $D \cap A$ contains $4 \cdot 4 = 16$ elements, while the intersections $A \cap C$ and $B \cap D$ each contain $4 \cdot 4 = 16$ elements. Each of the intersections of three of these sets contains 4 elements. The intersection of all four of the sets contains 4 elements.

The Inclusion-Exclusion Principle implies that the size of $A \cup B \cup C \cup D$ is $4 \cdot 64 - 6 \cdot 16 + 4 \cdot 4 - 4 = 172$. Thus the number of ways to plant the sections so that adjacent sections are not planted with incompatible crops is $256 - 172 = 84$.

19. A disk of radius 1 rolls all the way around the inside of a square of side length $s > 4$ and sweeps out a region of area A. A second disk of radius 1 rolls all the way around the outside of the same square and sweeps out a region of area $2A$. The value of s can be written as $a + \frac{b\pi}{c}$, where a, b, and c are positive integers and b and c are relatively prime. What is $a + b + c$?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

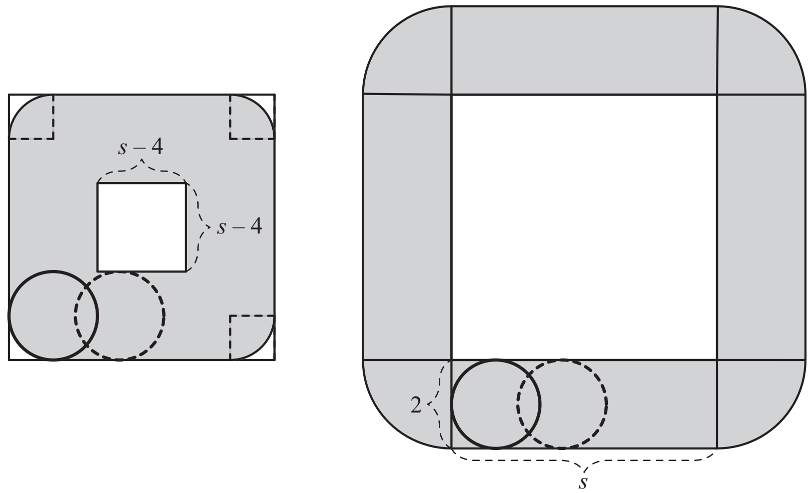
Answer (A): To obtain the region swept out by the first disk, remove a square of side length $s-4$ from the center of the original square and replace the 4 unit squares at the corners of the original square with 4 quarter-circles of radius 1. The area of this region is

$$s^2 - (s-4)^2 - 4 + \pi = 8s - 20 + \pi.$$

The region swept out by the second disk is the disjoint union of 4 rectangles, each with length s and width 2, and 4 quarter-circles of radius 2, so its area is $8s + 4\pi$. Therefore

$$8s + 4\pi = 2(8s - 20 + \pi).$$

Solving this equation gives $s = 5 + \frac{\pi}{4}$, so the requested sum is $5 + 1 + 4 = 10$.



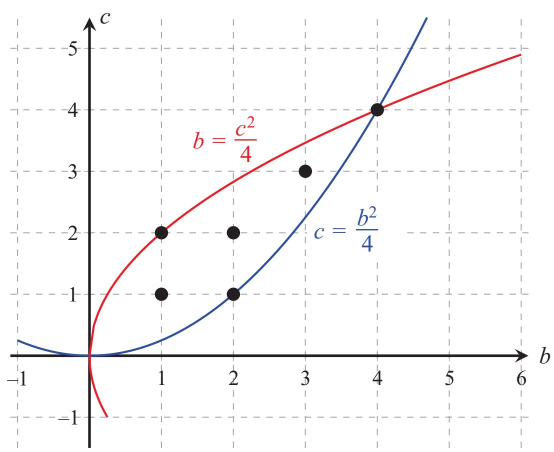
20. For how many ordered pairs (b, c) of positive integers does neither $x^2 + bx + c = 0$ nor $x^2 + cx + b = 0$ have two distinct real solutions?

(A) 4 **(B)** 6 **(C)** 8 **(D)** 12 **(E)** 16

Answer (B): The equation $x^2+bx+c = 0$ fails to have two distinct real solutions precisely when $b^2-4c \le 0$. Similarly the equation $x^2+cx+b = 0$ fails to have two distinct real solutions precisely when $c^2-4b \le 0$. Hence the given condition is satisfied if and only if $b^2 \le 4c$ and $c^2 \le 4b$, which can be written as $b^4 \le 16c^2 \le 64b$. Because $b > 0$, the inequality $b^4 \le 64b$ implies that $1 \le b \le 4$. If $b = 1$, then $1 \le 16c^2 \le 64$, so $c = 1$ or $c = 2$. If $b = 2$, then $16 \le 16c^2 \le 128$, so $c = 1$ or $c = 2$. If $b = 3$, then $81 \le 16c^2 \le 192$, so $c = 3$. If $b = 4$, then $256 \le 16c^2 \le 256$, so $c = 4$. The total number of ordered pairs (b, c) is $2 + 2 + 1 + 1 = 6$.

OR

As above, the required ordered pairs must satisfy $b^2 \le 4c$ and $c^2 \le 4b$, so they are on or inside both of the parabolas $c = \frac{b^2}{4}$ and $b = \frac{c^2}{4}$ in the bc-coordinate plane. Those parabolas intersect at $(0, 0)$ and $(4, 4)$, and the lattice points with positive coordinates on or inside both parabolas are $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$, so the number of ordered pairs is 6.



21. Each of 20 balls is tossed independently and at random into one of 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?

(A) 1 **(B)** 4 **(C)** 8 **(D)** 12 **(E)** 16

Answer (E): The requested ratio divides the number of ways to end up with a 3–4–4–4–5 distribution by the number of ways to end up with a 4–4–4–4–4 distribution. For either outcome, there are at least three bins with 4 balls each, leaving 8 balls to distribute into two bins. For a 3–5 split in the two bins, there are $5 \cdot 4 = 20$ ways to choose the bins, and $\binom{8}{3} = 56$ ways to choose 3 balls. For a 4–4 split in the two bins, there are $\binom{8}{4} = 70$ ways to choose 4 balls. The requested ratio is therefore $\frac{p}{q} = \frac{20 \cdot 56}{70} = 16$.

OR

The probabilities of the ball distribution follow the multinomial distribution. If there are n balls and k bins, then the probability that n_i balls end up in bin i for every i is given by

$$\frac{n!}{n_1! n_2! \cdots n_k!} \cdot p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k},$$

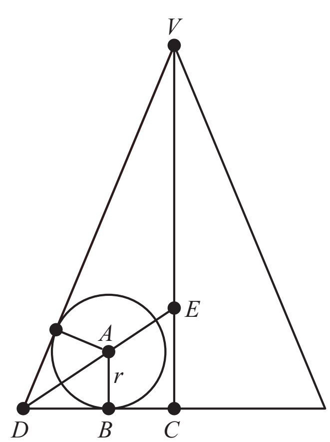
where p_i is the probability of any particular ball getting tossed into bin i, which equals $\frac{1}{k}$. There are $5 \cdot 4$ choices for which bin gets 3 balls and which bin gets 5 balls under the first scenario. The requested ratio is therefore

$$\frac{p}{q} = \frac{5 \cdot 4 \cdot \frac{20!}{3!4!4!4!5!} \cdot \left(\frac{1}{5}\right)^{20}}{\frac{20!}{4!4!4!4!4!} \cdot \left(\frac{1}{5}\right)^{20}} = 5 \cdot 4 \cdot \frac{4}{5} = 16.$$

22. Inside a right circular cone with base radius 5 and height 12 are three congruent spheres each with radius r. Each sphere is tangent to the other two spheres and also tangent to the base and side of the cone. What is r ?

(A) $\frac{3}{2}$ **(B)** $\frac{90 - 40\sqrt{3}}{11}$ **(C)** 2 **(D)** $\frac{144 - 25\sqrt{3}}{44}$ **(E)** $\frac{5}{2}$

Answer (B): Let C be the center of the base of the cone, A be the center of one of the spheres, B be the point where that sphere is tangent to the base of the cone, D be the point such that $\overline{CD}$ is a radius of the base of the cone containing B, V be the vertex of the cone, and E be the point on $\overline{CV}$ such that $\overline{ED}$ contains A. Let the radius of the sphere be $r = AB$.



Because the three spheres are mutually tangent, their centers are at the vertices of an equilateral triangle with side length $2r$. The line CV passes through the centroid of the equilateral triangle, so BC must be $\frac{2}{3}$ the altitude of that triangle, implying that $BC = \frac{2}{3} \cdot r\sqrt{3}$. The Pythagorean Theorem implies that $DV = \sqrt{CD^2 + CV^2} = 13$.

Observe that $\overline{DE}$ is the angle bisector of $\angle CDV$. By the Angle Bisector Theorem, $\frac{CD}{CE} = \frac{DV}{EV}$. Thus $\frac{5}{CE} = \frac{13}{12-CE}$ and solving gives $CE = \frac{10}{3}$. Because $\triangle DAB \sim \triangle DEC$, it follows that

$$BD = AB \cdot \frac{CD}{CE} = r \cdot \frac{5}{\left(\frac{10}{3}\right)} = \frac{3}{2}r.$$

Then $BC + BD = CD$ gives

$$\frac{2\sqrt{3}}{3}r + \frac{3}{2}r = 5.$$

Solving for r yields $r = \frac{90-40\sqrt{3}}{11}$.

23. For each positive integer n, let $f_1(n)$ be twice the number of positive integer divisors of n, and for $j \geq 2$, let $f_j(n) = f_1(f_{j-1}(n))$. For how many values of $n \leq 50$ is $f_{50}(n) = 12$?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

Answer (D): In the table below, p is a prime, and $p_1, p_2, \ldots$ denote distinct primes. Let $\tau(n)$ denote the number of positive divisors of n. If $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$, then

$$\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_r + 1).$$

Twelve important observations are found in the third column of the table.

Line	Factorization	Result	#
1	p^3	$f_1(n) = 8 = 2^3$, so 8 is a fixed point of f_1	2
2	$p_1 p_2$	$f_1(n) = 8$ and stays fixed at 8, by line 1	13
3	p^2	$f_1(n) = 6 = 2 \cdot 3$, which goes to 8 by line 2	4
4	p	$f_1(n) = 4 = 2^2$, which goes to 8 by line 3	15
5	1	$f_1(1) = 2$, which goes to 8 by line 4	1
6	$p_1^2 p_2$	$f_1(n) = 12 = 2^2 \cdot 3$, so **12** is a fixed point of f_1	**7**
7	p^4	$f_1(n) = 10 = 2 \cdot 5$, which goes to 8 by line 2	1
8	$p_1^3 p_2$	$f_1(n) = 16 = 2^4$, which goes to 8 by line 7	2
9	$p_1 p_2 p_3$	$f_1(n) = 16 = 2^4$, which goes to 8 by line 7	2
10	p^5	$f_1(n) = 12 = 2^2 \cdot 3$, which stays at **12** by line 6	**1**
11	$p_1^2 p_2^2$	$f_1(n) = 18 = 3^2 \cdot 2$, which goes to **12** by line 6	**1**
12	$p_1^4 p_2$	$f_1(n) = 20 = 2^2 \cdot 5$, which goes to **12** by line 6	**1**
			50

There are 7 numbers $n \leq 50$ whose factorization takes the form $p_1^2 p_2$, namely $12 = 2^2 \cdot 3$, $18 = 3^2 \cdot 2$, $20 = 2^2 \cdot 5$, $28 = 2^2 \cdot 7$, $44 = 2^2 \cdot 11$, $45 = 3^2 \cdot 5$, and $50 = 5^2 \cdot 2$. There is 1 number $n \leq 50$ whose factorization is p^5, namely $2^5 = 32$. There is 1 number $n \leq 50$ whose factorization is $p_1^2 p_2^2$, namely $2^2 \cdot 3^2 = 36$. There is 1 number $n \leq 50$ whose factorization is $p_1^4 p_2$, namely $2^4 \cdot 3 = 48$. In all, there are $7 + 1 + 1 + 1 = 10$ numbers $n \leq 50$ such that $f_{50}(n) = 12$.

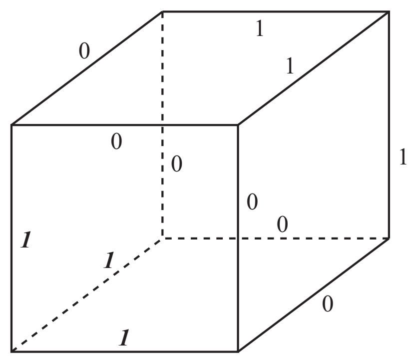
Note: Pair one divisor d of n with its complementary divisor $\frac{n}{d}$. Because at least one of d and $\frac{n}{d}$ is less than or equal to $\sqrt{n}$, it follows that $\tau(n) \leq 2\sqrt{n}$ for all n. Hence $f_1(n) \leq 4\sqrt{n}$, so $f_1(n) < n$ if $n > 16$. Indeed, after examining 13, 14, 15, and 16 it emerges that $f_1(12) = 12$, but $f_1(n) < n$ for all $n > 12$.

The estimate for $\tau(n)$ used here is convenient because it is easy to prove and it sufficed for the present purpose. With a little more work it can be shown that for every $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that $\tau(n) \leq C(\varepsilon)n^{\varepsilon}$ for all $n \geq 1$. Such an argument can be made to yield the best constant $C(\varepsilon)$ and an n for which equality is achieved. When $\varepsilon = \frac{1}{2}$, the best possible constant is $\sqrt{3}$, and equality is achieved when $n = 12$.

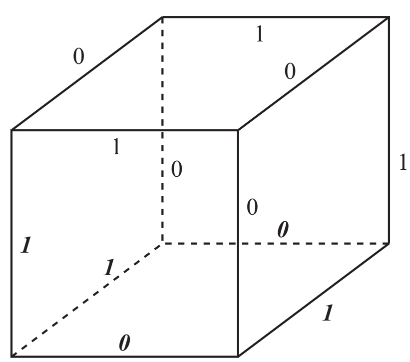
24. Each of the 12 edges of a cube is labeled 0 or 1. Two labelings are considered different even if one can be obtained from the other by a sequence of one or more rotations and/or reflections. For how many such labelings is the sum of the labels on the edges of each of the 6 faces of the cube equal to 2 ?

    (A) 8 **(B)** 10 **(C)** 12 **(D)** 16 **(E)** 20

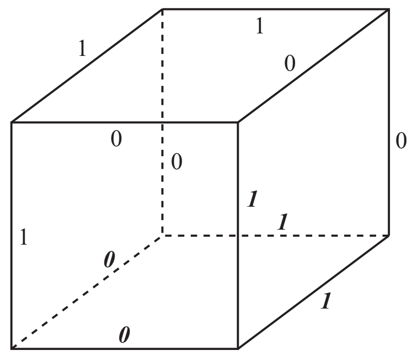
    Answer (E): First suppose that the three edges that share one particular vertex are all labeled 1. Then the rest of the labels are forced by the sum condition, and the figure below is obtained, up to rotation, where those first three edge labels are shown in bold italic font. Note that in this case there is a pair of vertices at the end of an interior diagonal, all of whose edges connected to it are labeled 1, with the remaining edges all labeled 0, no three of which mutually share a vertex. Because there are 4 interior diagonals, this gives <u>4</u> possible labelings. By symmetry there are another <u>4</u> possible labelings in which the roles of 0 and 1 are interchanged.



Otherwise, every vertex is connected to two edges labeled 1 and one edge labeled 0 or vice versa. Suppose that the labels of the edges on the bottom face of the cube are 0, 1, 0, 1 in that order. There are 2 possibilities, depending on which label is given to the front bottom edge. There are also 2 possibilities for the label of the left front edge. Once those five labels are determined, the rest of the labels are forced by the sum condition and the fact that no vertex is connected to edges with all the same label. See the figure below, in which the five mentioned labels are shown in bold italic font. This gives $2 \cdot 2 = \underline{4}$ labelings.



In the remaining case, the bottom face has labels 0, 0, 1, 1 in that order, which is 4 more cases. Say that the front and left bottom edges are labeled 0. Then the right front vertical edge can have either label, and once that label is chosen, again the rest of the labeling is forced. See the figure below, in which the five mentioned labels are again shown in bold italic font. This gives $4 \cdot 2 = \underline{8}$ more labelings.



Thus in all there are $4+4+4+8=20$ labelings satisfying the condition.

25. A quadratic polynomial $p(x)$ with real coefficients and leading coefficient 1 is called *disrespectful* if the equation $p(p(x))=0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?

(A) $\dfrac{5}{16}$ **(B)** $\dfrac{1}{2}$ **(C)** $\dfrac{5}{8}$ **(D)** 1 **(E)** $\dfrac{9}{8}$

Answer (A): Suppose $p(x)=(x-r)(x-s)$. Observe that $p(x)$ must have (two) real roots in order for $p(p(x))$ to have any roots at all. More specifically, if y is a root of $p(p(x))$, then $p(y)=r$ or $p(y)=s$. That is, the equations

$$(x-r)(x-s)-r=0 \quad \text{and} \quad (x-r)(x-s)-s=0$$

together must have exactly three real roots among them. It follows that one of these two quadratics, say $(x-r)(x-s)-r$, must have discriminant zero.

Expansion yields $x^2-(r+s)x+r(s-1)=0$, so the discriminant Δ of this quadratic must satisfy

$$0=\Delta=(r+s)^2-4r(s-1)=(r-s)^2+4r.$$

This implies that r is negative, say $r=-r_0$, and that $s=r\pm\sqrt{-4r}=-r_0\pm 2\sqrt{r_0}$. It follows that

$$r+s=2\left(-r_0\pm\sqrt{r_0}\right)\le 2\left(-r_0+\sqrt{r_0}\right)\le 2\cdot\frac{1}{4}=\frac{1}{2},$$

where the second inequality follows from the fact that $a-a^2\le\frac{1}{4}$ for all real numbers a. Thus $r=-\frac{1}{4}$ and $s=\frac{3}{4}$, which works. In turn, $\tilde{p}(x)=\left(x+\frac{1}{4}\right)\left(x-\frac{3}{4}\right)$ and $\tilde{p}(1)=\frac{5}{4}\cdot\frac{1}{4}=\frac{5}{16}$.

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