MAA AMC

American Mathematics Competitions

MAA American Mathematics Competitions

23rd Annual

AMC 10 A

FALL

Wednesday, November 10, 2021

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR COMPETITION MANAGER TELLS YOU TO BEGIN.
2. This is a 25-question multiple-choice competition. For each question, only one answer choice is correct.
3. Mark your answer to each problem on the answer sheet with a #2 pencil. Check blackened answers for accuracy and erase errors completely. Only answers that are properly marked on the answer sheet will be scored.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. Only blank scratch paper, blank graph paper, rulers, compasses, and erasers are allowed as aids. No calculators, smartwatches, phones, or computing devices are allowed. No problems on the competition will require the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the competition, your competition manager will ask you to record your name on the answer sheet.
8. You will have 75 minutes to complete the competition once your competition manager tells you to begin.
9. When you finish the competition, sign your name in the space provided on the answer sheet and complete the demographic information questions on the back of the answer sheet.

The MAA AMC office reserves the right to disqualify scores from a school if it determines that the rules or the required security procedures were not followed.

The publication, reproduction, or communication of the problems or solutions of this competition during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via phone, email, or digital media of any type during this period is a violation of the competition rules.

Students who score well on this AMC 10 will be invited to take the 40th annual American Invitational Mathematics Examination (AIME) on Tuesday, February 8, 2022, or Wednesday, February 16, 2022. More details about the AIME are on the back page of this test booklet.

1. What is the value of $\frac{(2112-2021)^2}{169}$?

   (A) 7 **(B)** 21 **(C)** 49 **(D)** 64 **(E)** 91

2. Menkara has a 4×6 index card. If she shortens the length of one side of this card by 1 inch, the card would have area 18 square inches. What would the area of the card be in square inches if instead she shortens the length of the other side by 1 inch?

   (A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

3. What is the maximum number of balls of clay with radius 2 that can completely fit inside a cube of side length 6 assuming that the balls can be reshaped but not compressed before they are packed in the cube?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

4. Mr. Lopez has a choice of two routes to get to work. Route A is 6 miles long, and his average speed along this route is 30 miles per hour. Route B is 5 miles long, and his average speed along this route is 40 miles per hour, except for a $\frac{1}{2}$-mile stretch in a school zone where his average speed is 20 miles per hour. By how many minutes is Route B quicker than Route A?

   (A) $2\frac{3}{4}$ **(B)** $3\frac{3}{4}$ **(C)** $4\frac{1}{2}$ **(D)** $5\frac{1}{2}$ **(E)** $6\frac{3}{4}$

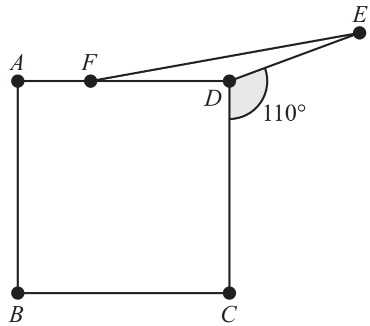
5. The six-digit number $\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}\,\underline{0}\,\underline{A}$ is prime for only one digit A. What is A ?

   (A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

6. Elmer the emu takes 44 equal strides to walk between consecutive telephone poles on a rural road. Oscar the ostrich can cover the same distance in 12 equal leaps. The telephone poles are evenly spaced, and the 41st pole along this road is exactly one mile (5280 feet) from the first pole. How much longer, in feet, is Oscar's leap than Elmer's stride?

   (A) 6 **(B)** 8 **(C)** 10 **(D)** 11 **(E)** 15

7. As shown in the figure below, point E lies in the opposite half-plane determined by line CD from point A so that $\angle CDE = 110^\circ$. Point F lies on $\overline{AD}$ so that $DE = DF$, and $ABCD$ is a square. What is the degree measure of $\angle AFE$?



(A) 160 **(B)** 164 **(C)** 166 **(D)** 170 **(E)** 174

8. A two-digit positive integer is said to be *cuddly* if it is equal to the sum of its nonzero tens digit and the square of its units digit. How many two-digit positive integers are cuddly?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

9. When a certain unfair die is rolled, an even number is 3 times as likely to appear as an odd number. The die is rolled twice. What is the probability that the sum of the numbers rolled is even?

(A) $\frac{3}{8}$ **(B)** $\frac{4}{9}$ **(C)** $\frac{5}{9}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{5}{8}$

10. A school has 100 students and 5 teachers. In the first period, each student is taking one class, and each teacher is teaching one class. The enrollments in the classes are 50, 20, 20, 5, and 5. Let t be the average value obtained if a teacher is picked at random and the number of students in their class is noted. Let s be the average value obtained if a student is picked at random and the number of students in their class, including that student, is noted. What is $t - s$?

(A) -18.5 **(B)** -13.5 **(C)** 0 **(D)** 13.5 **(E)** 18.5

11. Emily sees a ship traveling at a constant speed along a straight section of a river. She walks parallel to the riverbank at a uniform rate faster than the ship. She counts 210 equal steps walking from the back of the ship to the front. Walking in the opposite direction, she counts 42 steps of the same size from the front of the ship to the back. In terms of Emily's equal steps, what is the length of the ship?

(A) 70 **(B)** 84 **(C)** 98 **(D)** 105 **(E)** 126

12. The base-nine representation of the number N is $27{,}006{,}000{,}052_{\text{nine}}$. What is the remainder when N is divided by 5 ?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

13. Each of 6 balls is randomly and independently painted either black or white with equal probability. What is the probability that every ball is different in color from more than half of the other 5 balls?

(A) $\frac{1}{64}$ **(B)** $\frac{1}{6}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{5}{16}$ **(E)** $\frac{1}{2}$

14. How many ordered pairs (x, y) of real numbers satisfy the following system of equations?

$$x^2 + 3y = 9$$
$$(|x| + |y| - 4)^2 = 1$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 7

15. Isosceles triangle ABC has $AB = AC = 3\sqrt{6}$, and a circle with radius $5\sqrt{2}$ is tangent to line AB at B and to line AC at C. What is the area of the circle that passes through vertices A, B, and C ?

(A) 24π **(B)** 25π **(C)** 26π **(D)** 27π **(E)** 28π

16. The graph of $f(x) = |\lfloor x \rfloor| - |\lfloor 1 - x \rfloor|$ is symmetric about which of the following? (Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.)

(A) the y-axis **(B)** the line $x = 1$ **(C)** the origin

(D) the point $\left(\frac{1}{2}, 0\right)$ **(E)** the point $(1, 0)$

17. An architect is building a structure that will place vertical pillars at the vertices of regular hexagon $ABCDEF$, which is lying horizontally on the ground. The six pillars will hold up a flat solar panel that will not be parallel to the ground. The heights of the pillars at A, B, and C are 12, 9, and 10 meters, respectively. What is the height, in meters, of the pillar at E ?

    (A) 9 **(B)** $6\sqrt{3}$ **(C)** $8\sqrt{3}$ **(D)** 17 **(E)** $12\sqrt{3}$

18. A farmer's rectangular field is partitioned into a 2 by 2 grid of 4 rectangular sections as shown in the figure. In each section the farmer will plant one crop: corn, wheat, soybeans, or potatoes. The farmer does not want to grow corn and wheat in any two sections that share a border, and the farmer does not want to grow soybeans and potatoes in any two sections that share a border. Given these restrictions, in how many ways can the farmer choose crops to plant in each of the four sections of the field?

    (A) 12 **(B)** 64 **(C)** 84 **(D)** 90 **(E)** 144

19. A disk of radius 1 rolls all the way around the inside of a square of side length $s > 4$ and sweeps out a region of area A. A second disk of radius 1 rolls all the way around the outside of the same square and sweeps out a region of area $2A$. The value of s can be written as $a + \frac{b\pi}{c}$, where a, b, and c are positive integers and b and c are relatively prime. What is $a + b + c$?

    (A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

20. For how many ordered pairs (b, c) of positive integers does neither $x^2 + bx + c = 0$ nor $x^2 + cx + b = 0$ have two distinct real solutions?

    (A) 4 **(B)** 6 **(C)** 8 **(D)** 12 **(E)** 16

21. Each of 20 balls is tossed independently and at random into one of 5 bins. Let p be the probability that some bin ends up with 3 balls, another with 5 balls, and the other three with 4 balls each. Let q be the probability that every bin ends up with 4 balls. What is $\frac{p}{q}$?

    (A) 1 **(B)** 4 **(C)** 8 **(D)** 12 **(E)** 16

22. Inside a right circular cone with base radius 5 and height 12 are three congruent spheres each with radius r. Each sphere is tangent to the other two spheres and also tangent to the base and side of the cone. What is r ?

(A) $\frac{3}{2}$ **(B)** $\frac{90-40\sqrt{3}}{11}$ **(C)** 2 **(D)** $\frac{144-25\sqrt{3}}{44}$ **(E)** $\frac{5}{2}$

23. For each positive integer n, let $f_1(n)$ be twice the number of positive integer divisors of n, and for $j \geq 2$, let $f_j(n) = f_1(f_{j-1}(n))$. For how many values of $n \leq 50$ is $f_{50}(n) = 12$?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

24. Each of the 12 edges of a cube is labeled 0 or 1. Two labelings are considered different even if one can be obtained from the other by a sequence of one or more rotations and/or reflections. For how many such labelings is the sum of the labels on the edges of each of the 6 faces of the cube equal to 2 ?

(A) 8 **(B)** 10 **(C)** 12 **(D)** 16 **(E)** 20

25. A quadratic polynomial $p(x)$ with real coefficients and leading coefficient 1 is called *disrespectful* if the equation $p(p(x)) = 0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?

(A) $\frac{5}{16}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{5}{8}$ **(D)** 1 **(E)** $\frac{9}{8}$

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Scores and official competition solutions will be sent to your competition manager, who can share that information with you.

For more information about the MAA American Mathematics Competitions program and our other competitions, please visit maa.org/amc.

Questions and comments about this competition should be sent to

amcinfo@maa.org

or

MAA American Mathematics Competitions
P.O. Box 471
Annapolis Junction, MD 20701.

The problems and solutions for this AMC 10 A were prepared by the MAA AMC 10/12 Editorial Board under the direction of Azar Khosravani and Carl Yerger, co-Editors-in-Chief.

MAA Partner Organizations

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Akamai Foundation

Army Educational Outreach Program

Art of Problem Solving

AwesomeMathGirls.org

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MathWorks

Society for Industrial and Applied Mathematics

TBL Foundation

The D. E. Shaw Group

Tudor Investment Corporation

Two Sigma

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DO NOT OPEN until Wednesday, November 10, 2021

****Administration on an earlier date will disqualify your school's results.****

- All the information needed to administer this competition is contained in the AMC 10/12 Teacher's Manual. PLEASE READ THE MANUAL BEFORE WEDNESDAY, NOVEMBER 10, 2021.
- Answer sheets must be returned to the MAA AMC office within 24 hours of the competition administration. Use an overnight or 2-day shipping service, with a tracking number, to guarantee timely arrival of these answer sheets. FedEx, UPS, or USPS overnight are strongly recommended.
- The 40th annual American Invitational Mathematics Exam will be held on Tuesday, February 8, 2022, with an alternate date on Wednesday, February 16, 2022. It is a 15-question, 3-hour, integer-answer competition. Students who achieve a high score on the AMC 10 will be invited to participate. Top-scoring students on the AMC 10/12 and AIME will be selected to take the USA (Junior) Mathematical Olympiad.
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