MATHEMATICAL ASSOCIATION OF AMERICA
CELEBRATING A CENTURY OF ADVANCING MATHEMATICS

Solutions Pamphlet

American Mathematics Competitions

16th Annual

AMC 10 B

American Mathematics Contest 10B

Wednesday, February 25, 2015

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs conceptual, elementary vs advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Silvia Fernandez

1. **Answer (C):**

$$2-(-2)^{-2}=2-\frac{1}{(-2)^2}=2-\frac{1}{4}=\frac{7}{4}$$

2. **Answer (B):** The first two tasks together took 100 minutes—from 1:00 to 2:40. Therefore each task took 50 minutes. Marie began the third task at 2:40 and finished 50 minutes later, at 3:30 PM.

3. **Answer (A):** Let x be the integer Isaac wrote two times, and let y be the integer Isaac wrote three times. Then $2x+3y=100$. If $x=28$, then $3y=100-2\cdot 28=44$, and y cannot be an integer. Therefore $y=28$ and $2x=100-3\cdot 28=16$, so $x=8$.

4. **Answer (C):** After the first three siblings ate, there was $1-\frac{1}{5}-\frac{1}{3}-\frac{1}{4}=\frac{13}{60}$ of the pizza left for Dan to eat, so Dan ate more than $\frac{1}{5}=\frac{12}{60}$ but less than $\frac{1}{4}=\frac{15}{60}$ of the pizza. Because $\frac{1}{3}>\frac{1}{4}>\frac{13}{60}>\frac{1}{5}$, the order is Beth, Cyril, Dan, Alex.

5. **Answer (B):** Marta finished 6th, so Jack finished 5th. Therefore Todd finished 3rd and Rand finished 2nd. Because Hikmet was 6 places behind Rand, it was Hikmet who finished 8th. (David finished 10th.)

6. **Answer (E):** Marley plays basketball on Monday and golf on Wednesday. Because she cannot run three of the four consecutive days between Thursday and Sunday, she must run on Tuesday. From Thursday to Sunday she runs, swims, and plays tennis, but she cannot play tennis the day after running or swimming. So she must play tennis on Thursday. She must swim on Saturday, and run on Friday and Sunday, so that she does not run on consecutive days.

7. **Answer (A):**

$$\begin{aligned}((1\diamond 2)\diamond 3)-(1\diamond(2\diamond 3)) &= \left(\left(1-\frac{1}{2}\right)-\frac{1}{3}\right)-\left(1-\left(2-\frac{1}{3}\right)^{-1}\right)\\ &= \frac{1}{6}-\left(1-\frac{3}{5}\right)=\frac{1}{6}-\frac{2}{5}=-\frac{7}{30}\end{aligned}$$

8. **Answer (E):** The first rotation results in Figure 1, the reflection in Figure 2, and the half turn in Figure 3.

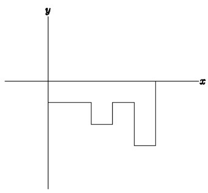


Figure 1

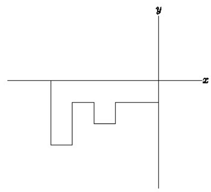


Figure 2

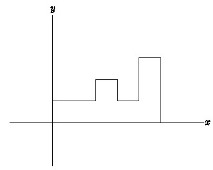


Figure 3

9. **Answer (B):** The shaded area is obtained by subtracting the area of the semicircle from the area of the quarter circle:

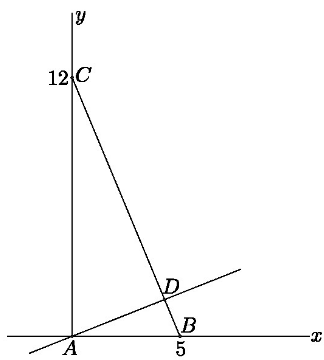
$$\frac{1}{4}\pi \cdot 3^2 - \frac{1}{2}\pi\left(\frac{3}{2}\right)^2 = \frac{9\pi}{4} - \frac{9\pi}{8} = \frac{9\pi}{8}.$$

10. **Answer (C):** There are 2014 negative integers strictly greater than -2015, and exactly half of them, or 1007, are odd. The product of an odd number of negative numbers is negative. Furthermore, because all factors are odd and some of them are multiples of 5, this product is an odd multiple of 5 and therefore has units digit 5.

11. **Answer (B):** There are four one-digit primes (2, 3, 5, and 7), which can be used to form $4^2 = 16$ two-digit numbers with prime digits. Of these two-digit numbers, only 23, 37, 53, and 73 are prime. So there are $4 + 16 = 20$ numbers less than 100 whose digits are prime, and $4 + 4 = 8$ of them are prime. The probability is $\frac{8}{20} = \frac{2}{5}$.

12. **Answer (A):** The circle intersects the line $y = -x$ at the points $A = (-5, 5)$ and $B = (5, -5)$. Segment $\overline{AB}$ is a chord of the circle and contains 11 points with integer coordinates.

13. **Answer (E):** Label the vertices of the triangle $A = (0, 0)$, $B = (5, 0)$, and $C = (0, 12)$. By the Pythagorean Theorem $BC = 13$. Two altitudes are 5 and 12. Let $\overline{AD}$ be the third altitude. The area of this triangle is 30, so $\frac{1}{2} \cdot AD \cdot BC = 30$. Therefore $AD = \frac{2 \cdot 30}{BC} = \frac{60}{13}$. The sum of the lengths of the altitudes is $5 + 12 + \frac{60}{13} = \frac{281}{13}$.



14. **Answer (D):** If $(x-a)(x-b)+(x-b)(x-c) = 0$, then $(x-b)(2x-(a+c)) = 0$, so the two roots are b and $\frac{a+c}{2}$. The maximum value of their sum is $9+\frac{8+7}{2} = 16.5$.

15. **Answer (B):** Let h be the number of horses and c be the number of cows. There are then $3h$ people, $9h$ ducks, and $4c$ sheep in Hamlet. The total population of Hamlet is $13h + 5c$, where h and c are whole numbers. A number N can be the population only if there exists a whole number value for h such that $N - 13h$ is a whole number multiple of 5. This is possible for all the provided numbers except 47, as follows: $41-13\cdot 2 = 5\cdot 3$, $59-13\cdot 3 = 5\cdot 4$, $61-13\cdot 2 = 5\cdot 7$, and $66 - 13 \cdot 2 = 5 \cdot 8$

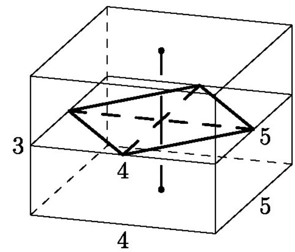
None of 47, $47 - 13 = 34$, $47 - 13 \cdot 2 = 21$, and $47 - 13 \cdot 3 = 8$ is a multiple of 5. Therefore 47 cannot be the population of Hamlet.

Note: In fact, 47 is the largest number that cannot be the population.

16. **Answer (C):** There are 9 assignments satisfying the condition: $(4, 2, 1)$, $(6, 2, 1)$, $(8, 2, 1)$, $(10, 2, 1)$, $(6, 3, 1)$, $(9, 3, 1)$, $(8, 4, 1)$, $(10, 5, 1)$, and $(8, 4, 2)$. There are $10 \cdot 9 \cdot 8 = 720$ possible assignments, so the probability is $\frac{9}{720} = \frac{1}{80}$.

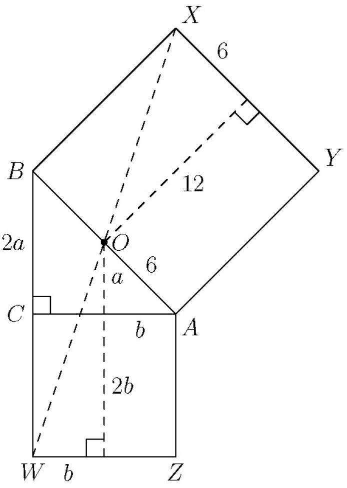
17. **Answer (B):** Consider the octahedron to be two pyramids whose base is a rhombus in the middle horizontal plane, as shown below. One pyramid points

up, the other down. The area of the base is the area of 4 right triangles with legs 2 and $\frac{5}{2}$, or 10. The altitude of each pyramid is half that of the prism or $\frac{3}{2}$. The volume of the octahedron is $2 \cdot \frac{1}{3} \cdot 10 \cdot \frac{3}{2} = 10$.

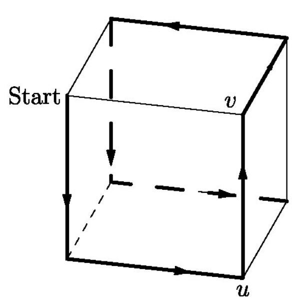


18. **Answer (D):** A coin can be tossed once, twice, or three times. View the problem as tossing each coin three times. If all three tosses are tails then the coin ends on a tail; however, if any of the three tosses is a head then the coin ends on a head (the subsequent tosses can be ignored). Thus each coin has a 7 out of 8 chance of landing on heads. Therefore the expected number of heads is $\frac{7}{8} \cdot 64 = 56$.

19. **Answer (C):** Let O be the center of the circle on which X, Y, Z, and W lie. Then O lies on the perpendicular bisectors of segments $\overline{XY}$ and $\overline{ZW}$, and $OX = OW$. Note that segments $\overline{XY}$ and $\overline{AB}$ have the same perpendicular bisector and segments $\overline{ZW}$ and $\overline{AC}$ have the same perpendicular bisector, from which it follows that O lies on the perpendicular bisectors of segments $\overline{AB}$ and $\overline{AC}$; that is, O is the circumcenter of $\triangle ABC$. Because $\angle C = 90^\circ$, O is the midpoint of hypotenuse $\overline{AB}$. Let $a = \frac{1}{2}BC$ and $b = \frac{1}{2}CA$. Then $a^2 + b^2 = 6^2$ and $12^2 + 6^2 = OX^2 = OW^2 = b^2 + (a + 2b)^2$. Solving these two equations simultaneously gives $a = b = 3\sqrt{2}$. Thus the perimeter of $\triangle ABC$ is $12+2a+2b = 12 + 12\sqrt{2}$.



20. **Answer (A):** The first two edges of Erin's crawl can be chosen in $3 \cdot 2 = 6$ ways. These edges share a unique face of the cube, called the initial face. At this point, Erin is standing at a vertex u and there is only one unvisited vertex v of the initial face. If v is not visited right after u, then Erin visits all vertices adjacent to v before v. This means that once Erin reaches v, she cannot continue her crawl to any unvisited vertex, and v cannot be her last visited vertex because v is adjacent to her starting point. Thus v must be visited right after u. There are only two ways to visit the remaining four vertices (clockwise or counterclockwise around the face opposite to the initial face) and exactly one of them cannot be followed by a return to the starting vertex. Therefore there are exactly 6 paths in all.

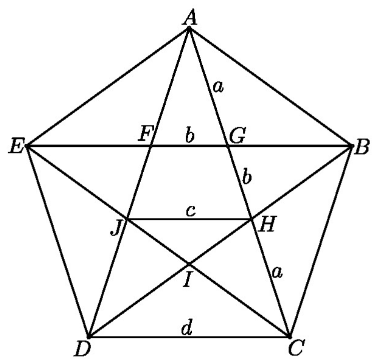


21. **Answer (D):** Assume that there are t steps in this staircase and it took Dash $d+1$ jumps. Then the possible values of t are $5d+1, 5d+2, 5d+3, 5d+4, 5d+5$. On the other hand, it took Cozy $d+20$ jumps, and $t = 2d+39$ or $t = 2d+40$.

There are 10 possible combinations but only 3 of them lead to integer values of d: $t = 5d + 3 = 2d + 39$, or $t = 5d + 1 = 2d + 40$, or $t = 5d + 4 = 2d + 40$. The possible values of t are 63, 66, and 64, and $s = 63 + 66 + 64 = 193$. The answer is $1 + 9 + 3 = 13$.

22. **Answer (D):** Triangles AGB and CHJ are isosceles and congruent, so $AG = HC = HJ = 1$. Triangles AFG and BGH are congruent, so $FG = GH$. Triangles AGF, AHJ, and ACD are similar, so $\frac{a}{b} = \frac{a+b}{c} = \frac{2a+b}{d}$.

Because $a = c = 1$, the first equation becomes $\frac{1}{b} = \frac{1+b}{1}$ or $b^2 + b - 1 = 0$, so $b = \frac{-1+\sqrt{5}}{2}$. Substituting this in the second equation gives $d = \frac{1+\sqrt{5}}{2}$, so $b + c + d = 1 + \sqrt{5}$.



23. **Answer (B):** Because there are ample factors of 2, it is enough to count the number of factors of 5. Let $f(n)$ be the number of factors of 5 in positive integers less than or equal to n. For n from 5 to 9, $f(n) = 1$. In order for $f(2n)$ to equal 3, $2n$ must be between 15 and 19, inclusive. Therefore $n = 8$ or $n = 9$. For n from 10 to 14, $f(n) = 2$. In order for $f(2n)$ to equal 6, $2n$ must be between 25 and 29, inclusive. Hence, $n = 13$ or $n = 14$. Thus the four smallest integers n that satisfy the specified condition are 8, 9, 13, and 14. Their sum is 44 and the sum of the digits of 44 is 8.

OR

In fact there are only 4 possible values of n. By Legendre's Theorem, if $n!$ ends in k zeros and $(2n)!$ ends in k' zeros, then

$$k = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots + \left\lfloor \frac{n}{5^j} \right\rfloor,$$

$$k' = \left\lfloor \frac{2n}{5} \right\rfloor + \left\lfloor \frac{2n}{5^2} \right\rfloor + \left\lfloor \frac{2n}{5^3} \right\rfloor + \cdots + \left\lfloor \frac{2n}{5^j} \right\rfloor + \left\lfloor \frac{2n}{5^{j+1}} \right\rfloor,$$

where j is the highest power of 5 not exceeding n, and thus the highest power of 5 not exceeding $2n$ is at most $j+1$. If x is a real number, then $\lfloor 2x \rfloor \le 2\lfloor x \rfloor + 1$. So $\left\lfloor \frac{2n}{5^i} \right\rfloor \le 2\left\lfloor \frac{n}{5^i} \right\rfloor + 1$ for each $1 \le i \le j+1$. Adding these inequalities yields $k' \le 2k + j + 1$. If $n \ge 15$, then $k > 2 + j - 1 = j + 1$ and $k' < 3k$. For $n = 13$ and $n = 14$, $k = 2$ and $k' = 5+1 = 6 = 3k$. For $n \le 12$, $k = \left\lfloor \frac{n}{5} \right\rfloor$ and $k' = \left\lfloor \frac{2n}{5} \right\rfloor$; in this case $k' = 3k$ only for $n = 8$ and $n = 9$. So $s = 8 + 9 + 13 + 14 = 44$ and the answer is $4 + 4 = 8$.

24. **Answer (D):** Note that for any natural number k, when Aaron reaches point $(k, -k)$, he will have just completed visiting all of the grid points within the square with vertices at $(k, -k)$, (k, k), $(-k, k)$, and $(-k, -k)$. Thus the point $(k, -k)$ is equal to $p_{(2k+1)^2-1}$. It follows that $p_{2024} = p_{(2\cdot 22+1)^2-1} = (22, -22)$. Because $2024 - 2015 = 9$, the point $p_{2015} = (22 - 9, -22) = (13, -22)$.

25. **Answer (B):** Because the volume and surface area are numerically equal, $abc = 2(ab+ac+bc)$. Rewriting the equation as $ab(c-6)+ac(b-6)+bc(a-6) = 0$ shows that $a \le 6$. The original equation can also be written as $(a-2)bc - 2ab - 2ac = 0$. Note that if $a = 2$, this becomes $b + c = 0$, and there are no solutions. Otherwise, multiplying both sides by $a - 2$ and adding $4a^2$ to both sides gives $[(a-2)b - 2a][(a-2)c - 2a] = 4a^2$. Consider the possible values of a.

$a = 1$: $(b+2)(c+2) = 4$
There are no solutions in positive integers.

$a = 3$: $(b-6)(c-6) = 36$
The 5 solutions for (b, c) are $(7, 42)$, $(8, 24)$, $(9, 18)$, $(10, 15)$, and $(12, 12)$.

$a = 4$: $(b-4)(c-4) = 16$
The 3 solutions for (b, c) are $(5, 20)$, $(6, 12)$, and $(8, 8)$.

$a = 5$: $(3b-10)(3c-10) = 100$
Each factor must be congruent to 2 modulo 3, so the possible pairs of factors are $(2, 50)$ and $(5, 20)$. The solutions for (b, c) are $(4, 20)$ and $(5, 10)$, but only $(5, 10)$ has $a \le b$.

$a = 6$: $(b-3)(c-3) = 9$
The solutions for (b, c) are $(4, 12)$ and $(6, 6)$, but only $(6, 6)$ has $a \le b$.

Thus in all there are 10 ordered triples (a, b, c): $(3, 7, 42)$, $(3, 8, 24)$, $(3, 9, 18)$, $(3, 10, 15)$, $(3, 12, 12)$, $(4, 5, 20)$, $(4, 6, 12)$, $(4, 8, 8)$, $(5, 5, 10)$, and $(6, 6, 6)$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Tom Butts, Steve Dunbar, Marta Eso, Zuming Feng, Silvia Fernandez, Charles Garner, Peter Gilchrist, Jerry Grossman, Joe Kennedy, Dan Kennedy, Cap Khoury, Steve Miller, David Wells, and Carl Yerger.

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